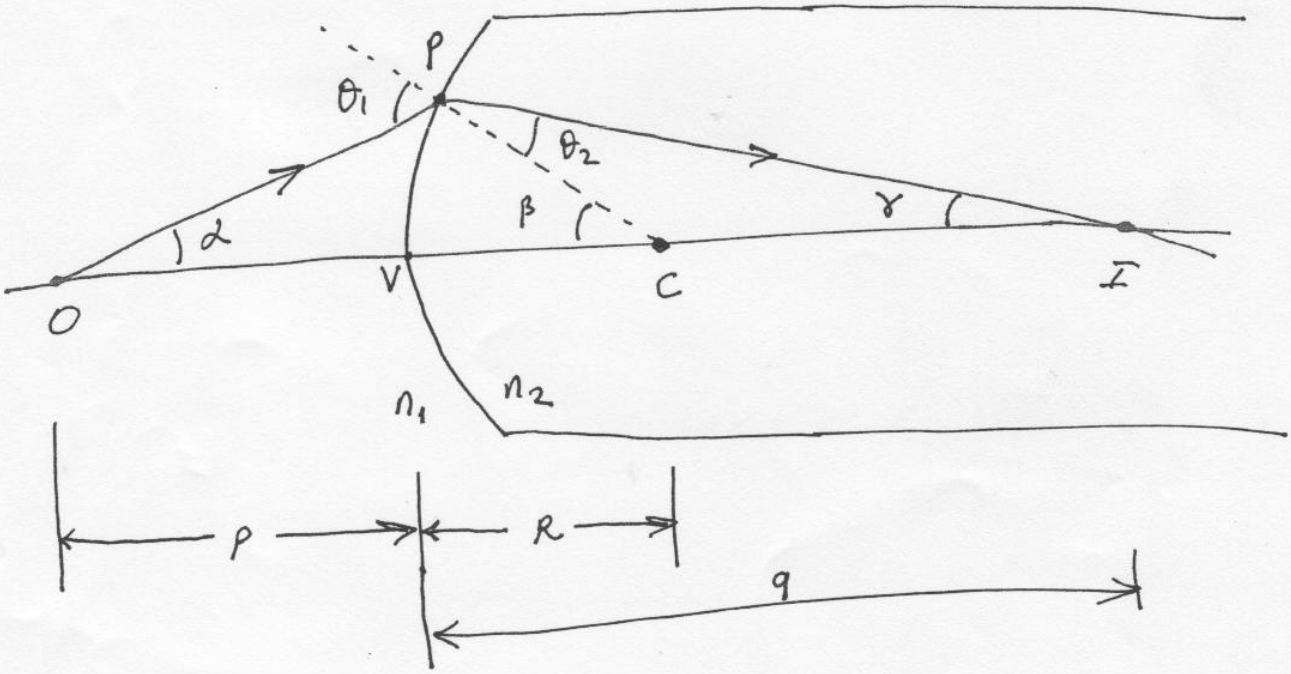
CHAPTER 4 THIN LENSES

4.1 Introduction

Lenses are commonly used to form images by refraction in optical instruments such as cameras, telescopes and microscopes. In this chapter we describe how images are formed by thin lenses.

4.2 Spherical Refraction surfaces



The image of point O is formed at point I.

Snell's law at point P:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

and

$$\theta_1 = \alpha + \beta$$
$$\beta = \theta_2 + \gamma$$

If we consider paraxial rays $\Rightarrow$ α is small

then β, γ, θ_1 and θ_2 will be small.

Using approx. $\sin x \approx x$ we obtain

$$n_1 \theta_1 \approx n_2 \theta_2$$

Therefore:

$$\theta_1 = \alpha + \beta$$

$$\beta = \theta_2 + \gamma = \frac{n_1}{n_2}\theta_1 + \gamma$$

Eliminate θ_1 from these two equations!

$$n_1 \alpha + n_2 \gamma = (n_2 - n_1)\beta$$

Radian measure of angles:

$$\alpha \approx \frac{|PV|}{p} \quad ; \quad \beta = \frac{|PV|}{R} \quad ; \quad \gamma \approx \frac{|PV|}{q}$$

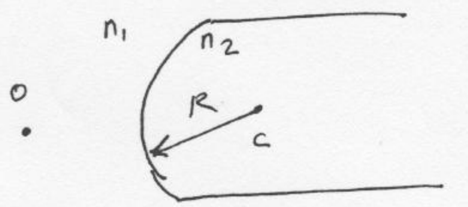
Substitution yields:

$$\boxed{\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}}$$

Sign conversion for refracting surfaces

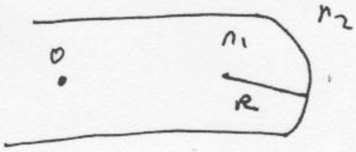
Quantity	Positive when	Negative when
p	O is in front of surface (real object)	O is in back of surface (virtual obj
q	I is in back of surface (real image)	I is in front of surface (virtual img
h'	Image is upright	Image is inverted
R	C is in back of surface	C is in front of surface

p	q	R
+	+	+
+	−	−
+	−	−



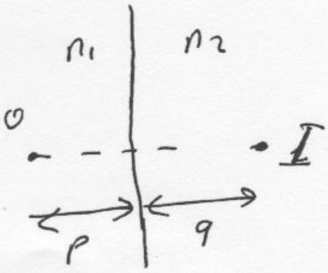
$n_1 < n_2$

$n_1 < n_2$



$n_1 > 1$

Note that if $R \to \infty$ (flat surface)

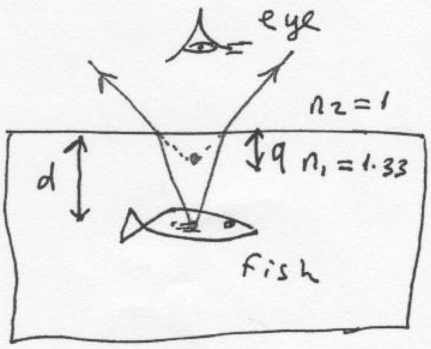


$$\frac{n_1}{p} + \frac{n_2}{q} = 0$$

$$q = -\frac{n_2}{n_1} p$$

EXAMPLE 1

A fish is swimming at a dept $d = 1\,m$. what is the apparent depth of the fish?

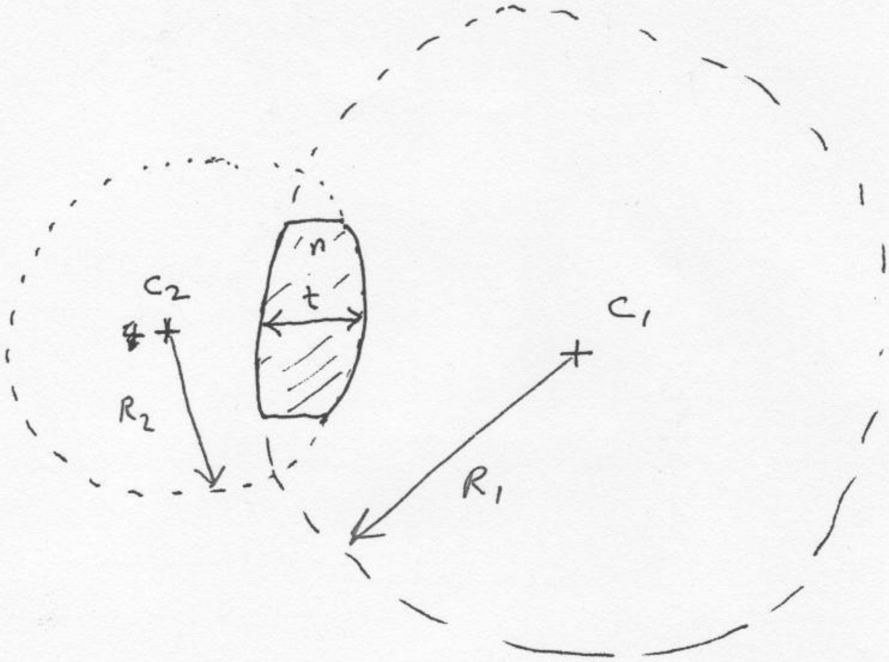


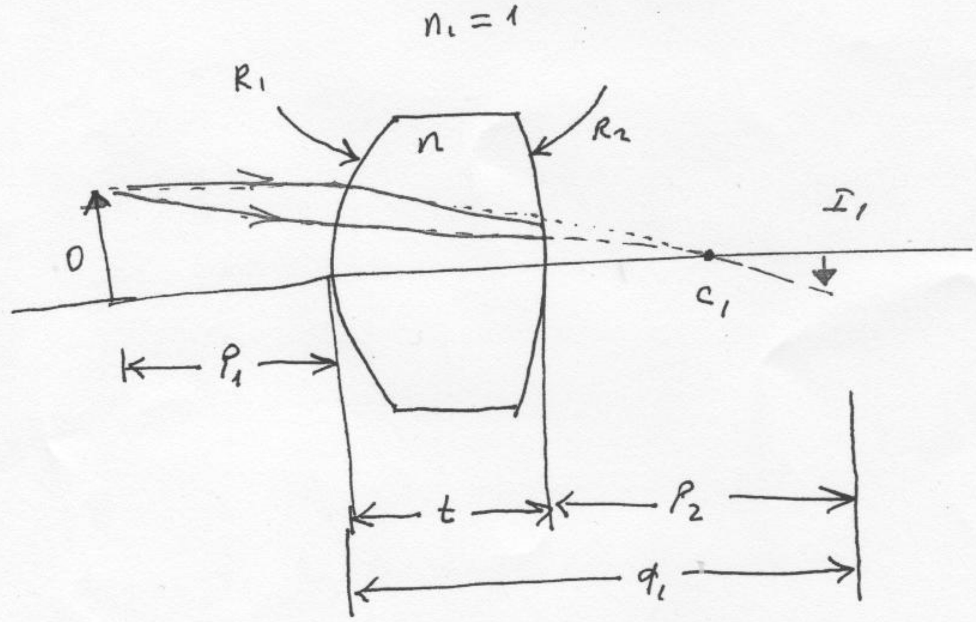
SOLUTION

$$q = -\frac{n_2}{n_1} p = -\frac{1}{1.33}(1) = -0.75\,m$$

image is virtual since $q < 0$.

4.3 Thin Lenses





*** The image formed by one refracting surface serves as the object for the second surface ***

- Image(I_1) formed by surface 1 ($n_1 = 1 \quad n_2 = n$)

$$\frac{1}{p_1} + \frac{n}{q_1} = \frac{n-1}{R_1} \qquad (\Delta)$$

- image of I_1 formed by surface 2 ($n_1 = n \quad n_2 = 1$)

$$\frac{n}{p_2} + \frac{1}{q_2} = \frac{1-n}{R_2} \qquad (*)$$

we make this switch in index since rays approaching surface 2 are "in the material of the lens".

- From figure $\quad p_2 = -q_1 + t \leftarrow$ why? see sign conversion table

for a thin lens $t \to 0$ and $p_2 = -q_1$. Eqn (*) becomes

$$-\frac{n}{q_1} + \frac{1}{q_2} = \frac{1-n}{R_2} \qquad (\square)$$

Adding (Δ) and ($\square$) yields

$$\frac{1}{p_1} + \frac{1}{q_1} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

for a thin lens we omit subsripts

$$\frac{1}{p} + \frac{1}{q} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

if $p \to \infty \Rightarrow q \to f$

$$\boxed{\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]}$$

lens maker's equation

As in mirrors

$$\boxed{\frac{1}{p} + \frac{1}{q} = \frac{1}{f}}$$

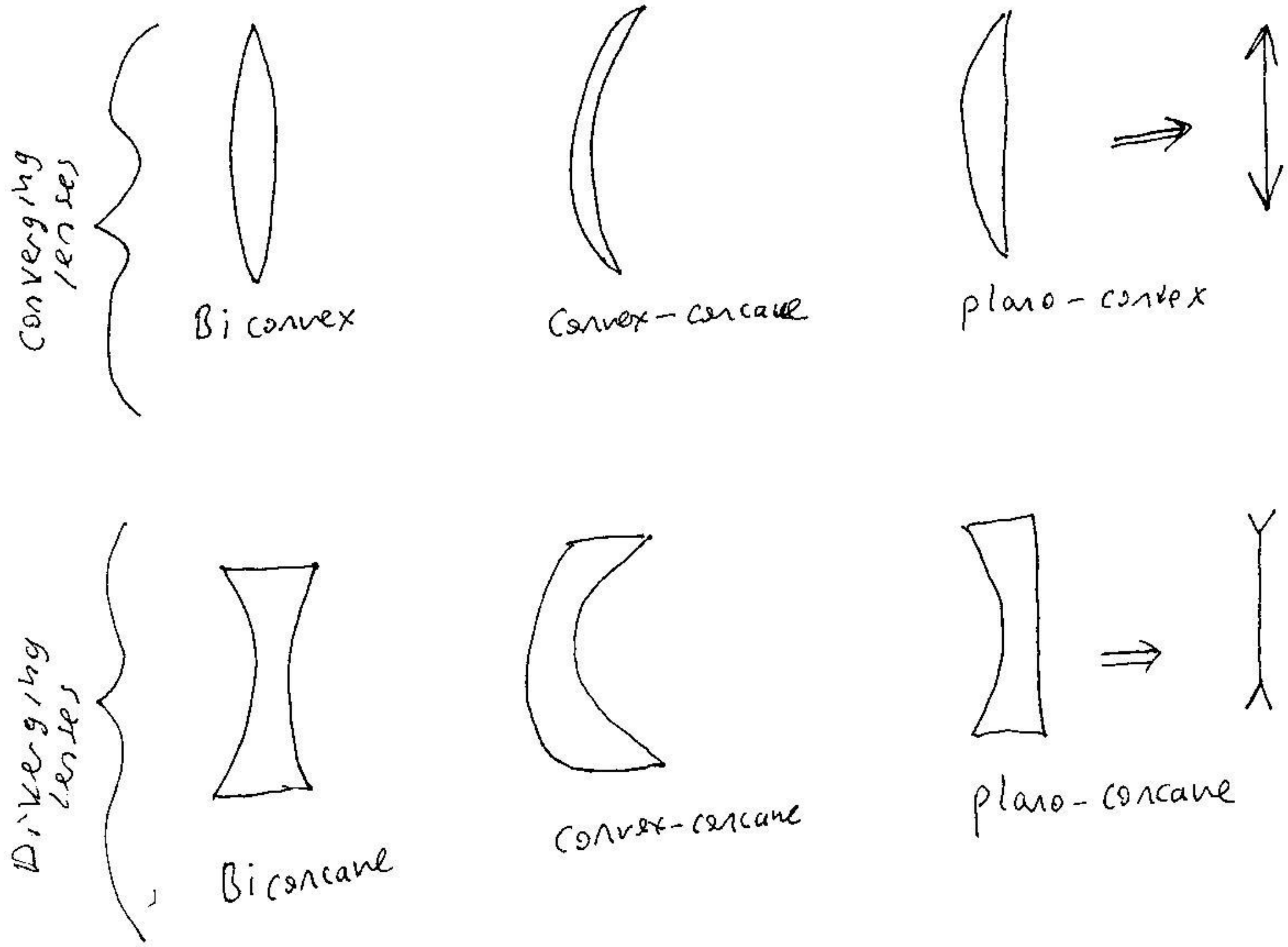
Gauss' Lens equation.

magnification

$$\boxed{m = \frac{h'}{h} = -\frac{q}{p}}$$

LAST 3 EQNS ARE VALID FOR BOTH LENSES

Lens shapes



Note that

- if the lens has a refractive index n_2 and its surrounding medium has n_1 then

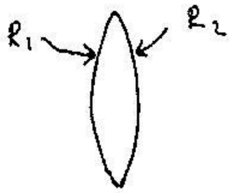
$$\frac{1}{f} = \frac{n_2 - n_1}{n_1}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

- if we don't neglect the thickness t of the lens:

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)t}{n R_1 R_2}\right]$$

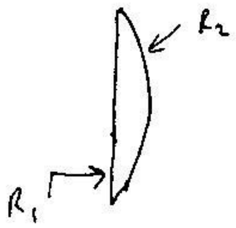
lens is in air.

Sign conversion for lens makers Eqn (in air)



$R_1 > 0$ and $R_2 < 0$

$\frac{1}{R_1} - \frac{1}{R_2} > 0 \Rightarrow f > 0$

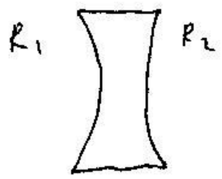


$R_1 = \infty$

$R_2 < 0$

$\frac{1}{R_1} - \frac{1}{R_2} > 0 \Rightarrow f > 0$

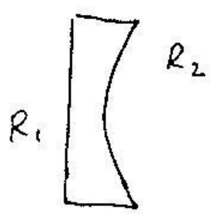
converging lens



$R_1 < 0$

$R_2 > 0$

$\frac{1}{R_1} - \frac{1}{R_2} < 0 \Rightarrow f < 0$



$R_1 = \infty$

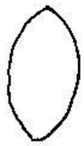
$R_2 > 0$

$\frac{1}{R_1} - \frac{1}{R_2} < 0 \Rightarrow f < 0$

diverging lens

<u>Example</u> A bi-convex lens of 50 mm focal length is to be made of glass ($n = 1.52$). One radius of curvature is to be twice that of the other. What are two radius of curvatures?

Lensmaker's equation:

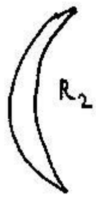
$R = R_1$  $R_2 = 2R$

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

$$= (n-1)\left[\frac{1}{R} - \frac{1}{-2R}\right]$$

$$\Rightarrow R = \frac{3(n-1)f}{2} = \frac{3\,(1.52-1)(50\text{ mm})}{2} \doteq 39\text{ mm} \begin{cases} R_1 = 39\text{ mm} \\ R_2 = -78\text{ mm} \end{cases}$$

<u>Example</u> A contact lens is made of plastic with an index of refraction of $n_{lens} = 1.5$. The lens has an outer radius of curvature $R_1 = +2$ cm and an inner radius of curvature of $R_2 = +2.5$ cm. Find the focal length of the lens (a) in air and (b) in water ($n_{water} = 1.33$)

R_1  R_2

(a) lensmakers' formula:

$$\frac{1}{f} = \left(\frac{n_{lens}}{n_{air}} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= \left(\frac{1.5}{1.0} - 1\right)\left(\frac{1}{2\text{cm}} - \frac{1}{2.5\text{cm}}\right)$$

$$= 0.05\text{ cm}^{-1} \rightarrow f = \frac{1}{0.05} = \underline{20\text{ cm.}}$$

(b)
$$\frac{1}{f} = \left(\frac{n_{lens}}{n_{water}} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= \left(\frac{1.5}{1.33} - 1\right)\left(\frac{1}{2} - \frac{1}{2.5}\right)$$

$$= 0.0125\text{ cm}^{-1} \rightarrow f = \frac{1}{0.0125} = \underline{80\text{ cm}}$$

4.5 Power of a lens

To calculate the power of a lens, we use the relationship

$$P = \frac{1}{f}$$

In this formula, focal lengths are usually measured in meters and P in Dioptre (D)

$$1\ D = 1\ m^{-1}$$

<u>Example</u> what is the power of the lens in the previous example?

(a) $P = \frac{1}{f} = \frac{1}{0.2\ m} = 5\ D$

(b) $P = \frac{1}{f} = \frac{1}{0.8\ m} = 1.25\ D$

4.4 Ray Diagrams

<u>Convex lens</u>

<u>Concave lens</u>

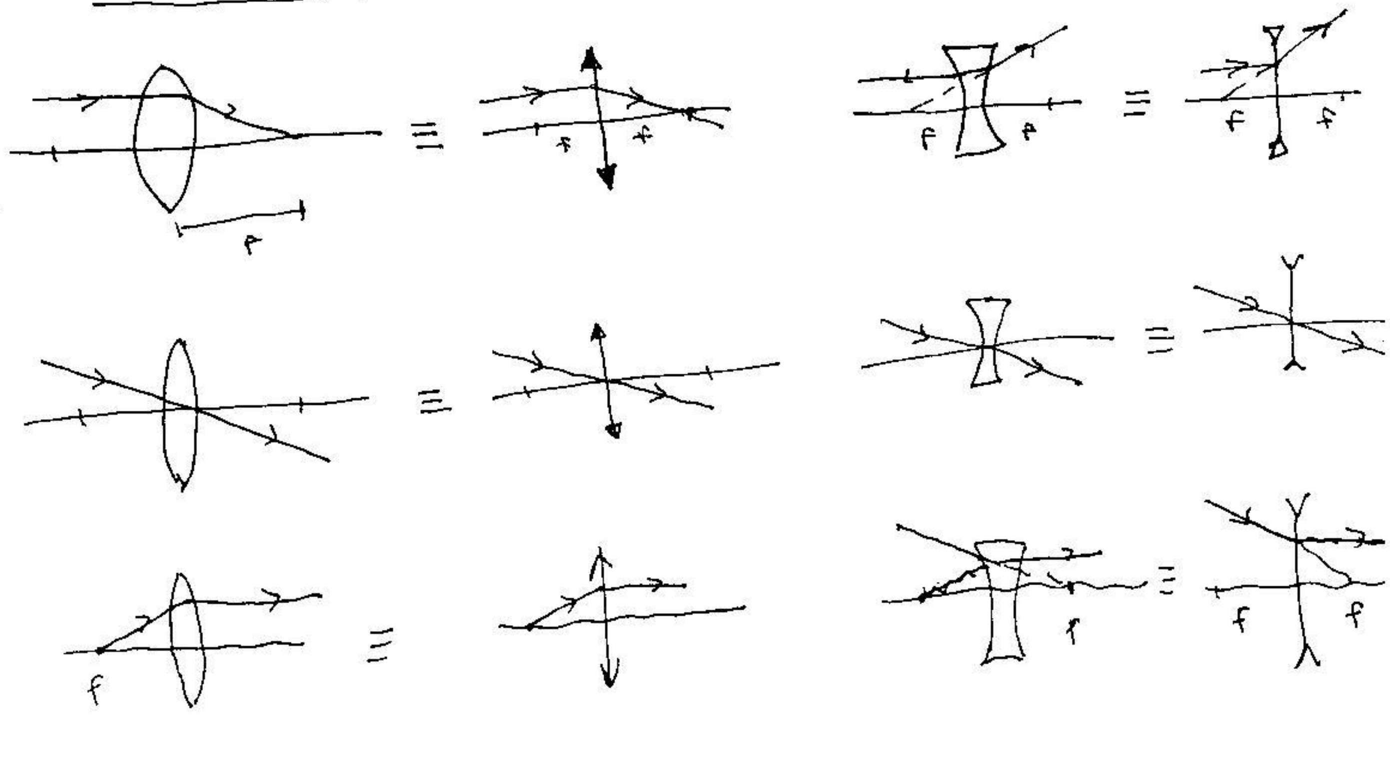
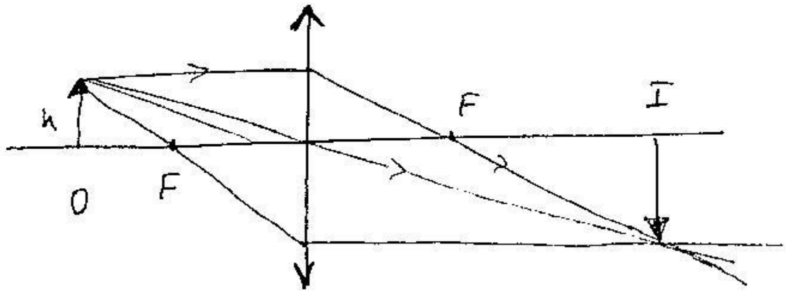
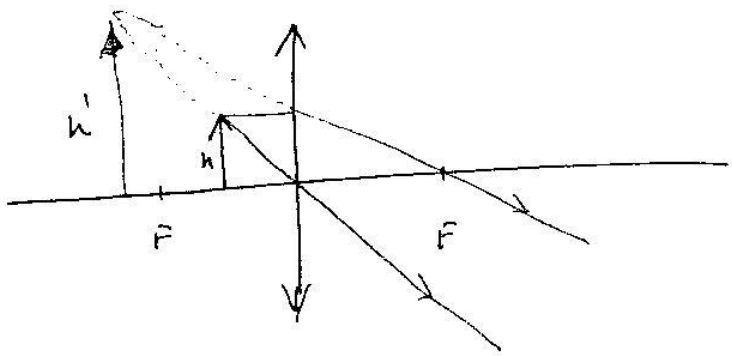


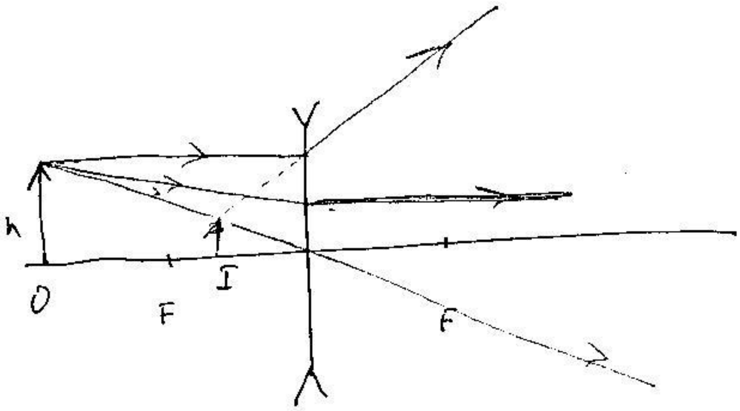
Image Formation



converging lens
real image



converging lens
virtual image



Diverging lens
always produces
a virtual image

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$m = \frac{h'}{h} = -\frac{q}{p}$$

f : + for convex lens
– for concave lens

p : + for real object
– for virtual object

q : + for real image
– for virtual image

EXAMPLE A converging lens has a focal length of $f = 20$ cm. Find the position and magnification of ~~the~~ object at a distance (a) 50 cm and (b) 10 cm from the lens.

SOLUTION

(a) $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$

$\frac{1}{20} = \frac{1}{50} + \frac{1}{q}$

$q = 33.3$ cm

$m = -\frac{q}{p} = -\frac{33.3}{50} =$

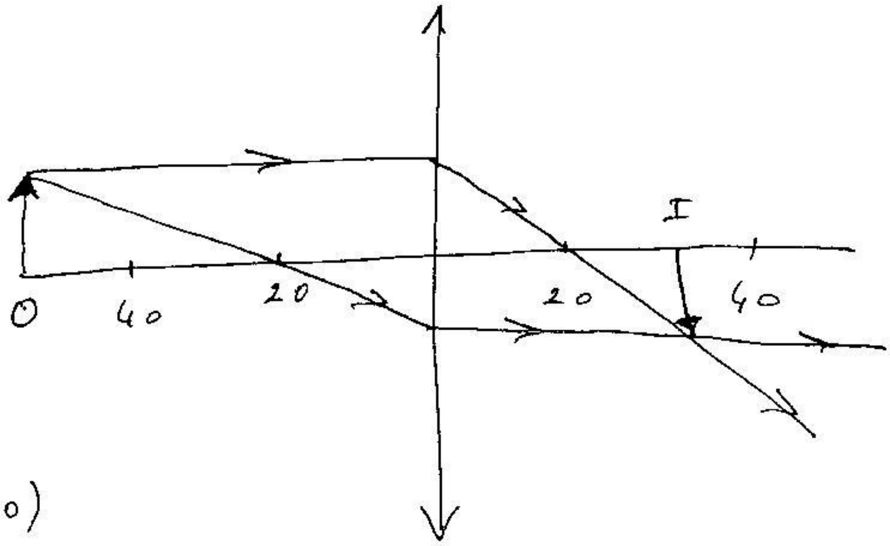


image is real ($q > 0$)
is inverted ($m < 0$)
is smaller ($|m| < 1$)

(b) $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$

$\frac{1}{20} = \frac{1}{10} + \frac{1}{q}$

$q = -20$ cm

$m = -\frac{q}{p} = -\frac{-20}{10} = +2$

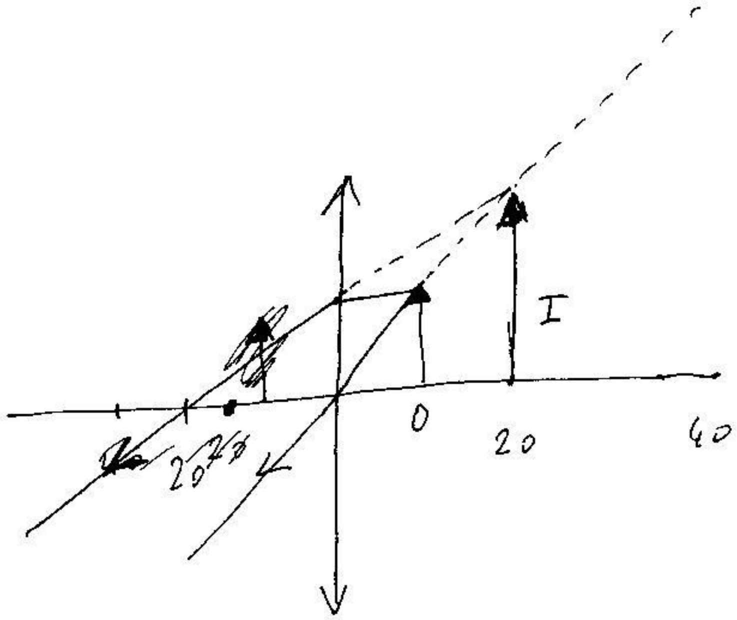


image is virtual ($q < 0$)
is upright ($m > 0$)
is larger ($|m| > 1$)

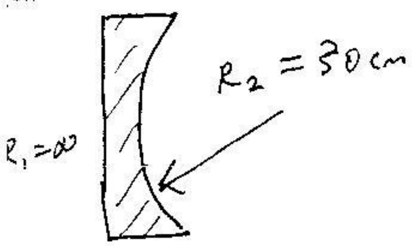
EXAMPLE A plano-convex les has an index of refraction and its ~~the~~ spherical face has a radius of curvature of +30 cm.

(a) Find the focal length of the lens in air.

(b) Find the position of the image and the magnification for the object $q = 150$ cm from the lens.

SOLUTION

a)



$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

$$= (1.5-1)\left[\frac{1}{\infty} - \frac{1}{+30}\right]$$

$$f = -60 \text{ cm}$$

(b) $$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$\frac{1}{-60} = \frac{1}{150} + \frac{1}{q}$$

$$q = -42.9 \text{ cm}$$

$$m = -\frac{q}{p} = -\frac{-42.9}{150}$$

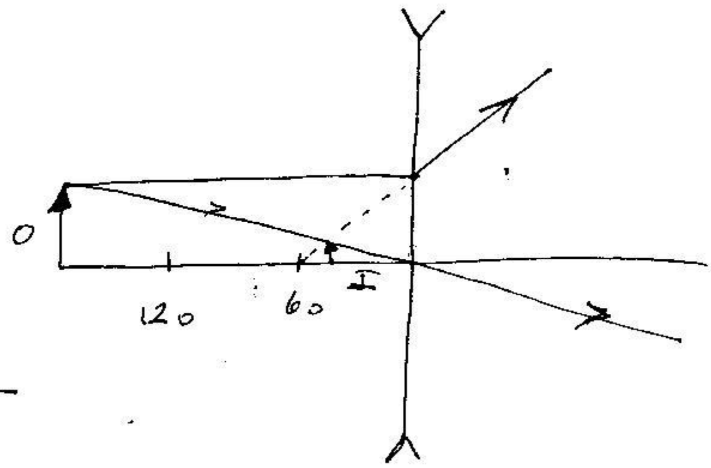
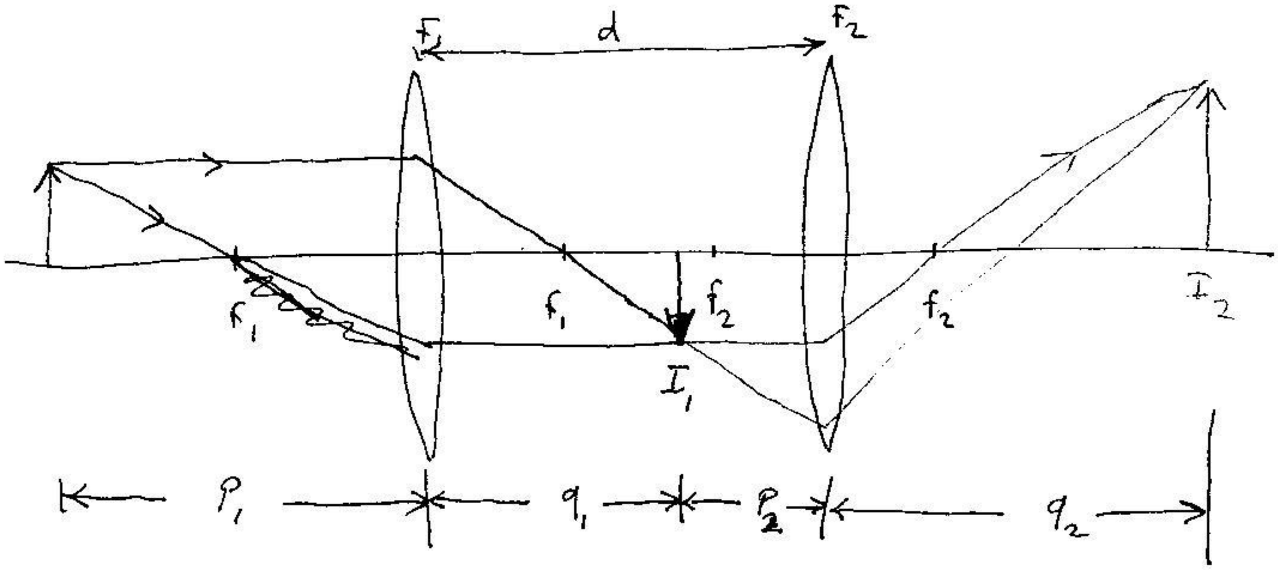


image is virtual $(q<0)$

is upright $(m>0)$

is smaller $(|m|<1)$

4.5 Lens Combinations



- $\dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{f_1}$; $m_1 = -\dfrac{q_1}{p_1}$

- $\dfrac{1}{p_2} + \dfrac{1}{q_2} = \dfrac{1}{f_2}$; $m_2 = -\dfrac{q_2}{p_2}$

$$p_2 = d - q_1$$

$$\therefore \quad q_2 = \frac{f_2 d - \dfrac{f_2 - p_1 f_1}{p_1 - f_1}}{d - f_2 - \dfrac{p_1 f_1}{p_1 - f_1}}$$

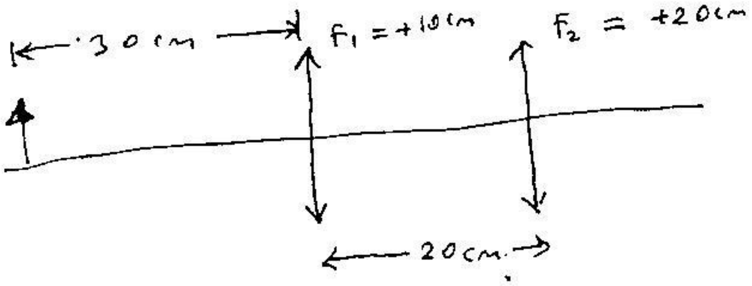
focal length of the system if $p_1 \to \infty \Rightarrow q_2 \to f_{system}$

$$\lim_{p_1 \to \infty} q_2 = f_{sys} = \frac{f_2 (d - f_1)}{d - (f_1 + f_2)}$$ (Back focal length)

magnification of the system.

$$m_{sys} = m_1 m_2$$

EXAMPLE Find the position and magnification of the final image produced by the following system:



SOLUTION

- $$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{f_1}$$
  $$\frac{1}{30} + \frac{1}{q_1} = \frac{1}{10}$$
  $$\boxed{q_1 = +15 \text{ cm}}$$ ← real image

  $$m_1 = -\frac{q_1}{p_1} = -\frac{15}{30} = -0.5$$ ← inverted

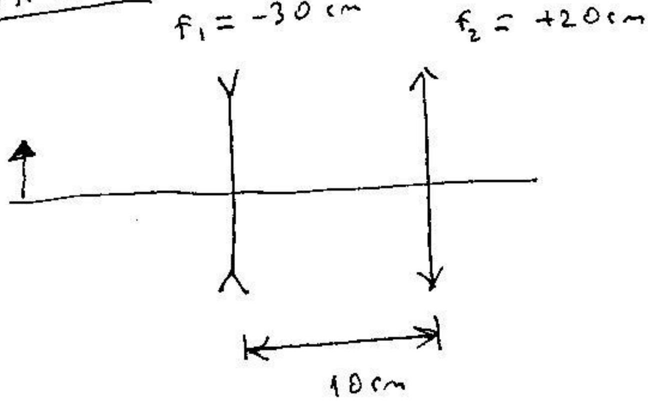
- $$\frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{f_2}$$
  $$\frac{1}{20-15} + \frac{1}{q_2} = \frac{1}{20}$$
  $$\boxed{q_2 = -6.67 \text{ cm}}$$ ← virtual image

  $$m_2 = -\frac{q_2}{p_2} = -\frac{-6.67}{5} = +1.33$$ ← upright

overall magnification:

$$m = m_1 m_2 = (-0.5)(+1.33) = -0.67$$

EXAMPLE



Find the focal length of the system.

SOLUTION

$$B.F.L = f = \frac{f_2(d - f_1)}{d - (f_1 + f_2)} = \frac{20(10 - (-30))}{10 - (-30 + 20)} = 40\text{ cm}$$