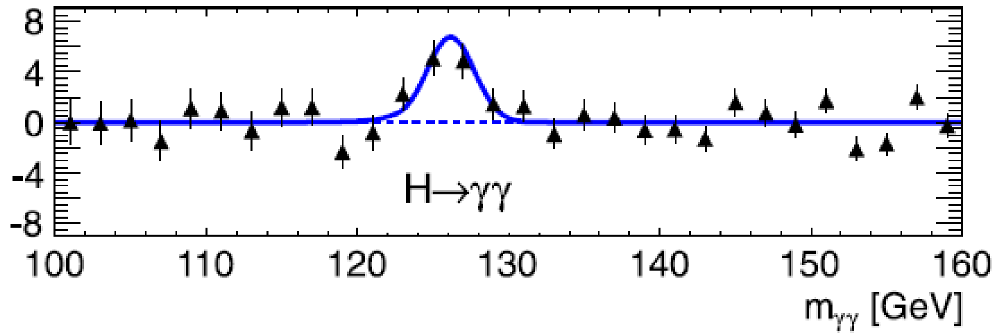
EP122
Measurement Techniques and Calibration

Topic 2

Error Propagation



http://www.gantep.edu.tr/~bingul/ep122

Department of
Engineering Physics

University of Gaziantep

Mar 2015

Mean and Standard Deviation

For a set of *N* measurements $\{x_1, x_2, x_3, ..., x_N\}$:

The arithmetic mean: $$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad (1)$$

The standard deviation: $$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2} \qquad (2)$$

The standard error: $$\sigma_E = \frac{\sigma}{\sqrt{N}} \qquad (3)$$

- Note that the square of the standard deviation is known as <u>variance</u>.

$$Variance \equiv \sigma^2$$

- Final result obtained from this kind of measurement should be reported in the form of:

$$\bar{x} \pm \sigma_E$$

Example 1

Consider the data of 10 different measurements for the mass density in g/cm^3 of a liquid.

d = { 1.10, 1.12, 1.09, 1.09, 1.07, 1.14, 1.11, 1.16, 1.07, 1.08}

The mean, standard deviation, variance and standard error of the measurement are as follows:

$$\overline{x} = (1.10 + 1.12 + 1.09 + 1.09 + 1.07 + 1.14 + 1.11 + 1.16 + 1.07 + 1.08)/10 = 1.103\ \mathrm{g/cm^3}$$

$$\sigma = \sqrt{[(1.10 - 1.103)^2 + (1.12 - 1.103)^2 + \cdots + (1.08 - 1.103)^2]/9} = 0.030\ \mathrm{g/cm^3}$$

$$\sigma^2 = 0.0009\ (\mathrm{g/cm^3})^2$$

$$\sigma_E = 0.03/\sqrt{10} = 0.009\ \mathrm{g/cm^3}$$

The result of the measurement is reported as $d = 1.103 \pm 0.009\ \mathrm{g/cm^3}$ or $1.103(9)\ \mathrm{g/cm^3}$.

Error Propagation (=Hata Birikimi)

In a typical experiment, one is seldom interested in taking data of a single quantity. More often, the data are processed through multiplication, addition or other functional manipulation to get final result. Experimental measurements have uncertainties due to measurement limitations which propagate to the combination of variables in a function.

In statistical data analysis, the propagation of error (or propagation of uncertainty) is the effect of measurement uncertainties (or errors) on the uncertainty of a function based on them.

For a function of one variable, f(x), if x has a measurment error σ_x, then the associated error σ_f is computed from:

$$\sigma_f = \left| \frac{\partial f}{\partial x} \right| \sigma_x \qquad (4)$$

Specific cases:

$$\sigma(x^2) = 2x\sigma_x \quad \text{or} \quad \frac{\sigma(x^2)}{x^2} = 2\frac{\sigma_x}{x}$$

$$\sigma(x^n) = nx^{n-1}\sigma_x \quad \text{or} \quad \frac{\sigma(x^n)}{x^n} = n\frac{\sigma_x}{x}$$

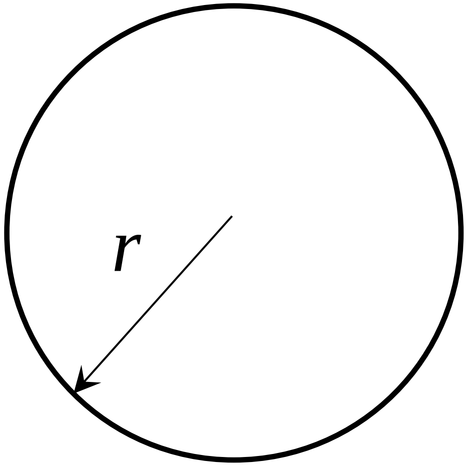
$$\sigma(\sin x) = \cos x \, \sigma_x$$

$$\sigma(\ln x) = \frac{1}{x}\sigma_x$$

Example 2

The radius of a circle is measured as

$$r = 2.5 \pm 0.3\,\mathrm{cm}$$



Calculate the area of the circle and its uncertainty.

Answer: $A = 19.6 \pm 4.7\ \mathrm{cm}^2$

In general, if x_1, x_2, …, x_n are independent variables having associated errors (standard deviations)

$$\sigma_1, \sigma_2, \ldots, \sigma_n$$

then the standard deviation for any quantity of the form

$$f = f\,(x_1, x_2, \ldots, x_n)$$

derived from these errors can be calculated from:

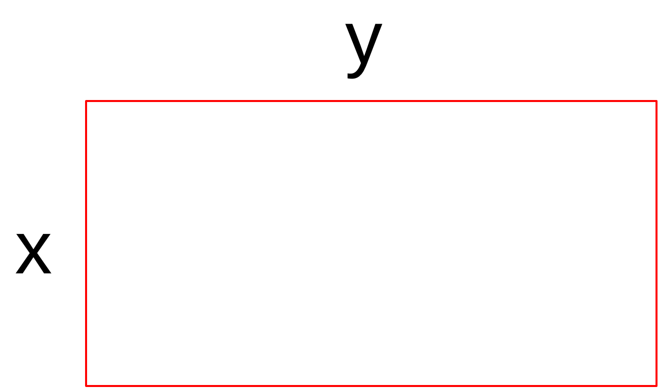
$$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_n^2 \qquad (5)$$

Errors of simple functions after application of Eqn. (5).

Function	Derivative(s)	Variance	Standard deviation
$f = kx$; $k \varepsilon \mathrm{R}$	$\frac{\partial f}{\partial x} = k$	$\sigma_f^2 = k^2 \sigma_x^2$	$\sigma_f = k\sigma_x$
$f = x + y$	$\frac{\partial f}{\partial x} = 1$ and $\frac{\partial f}{\partial y} = 1$	$\sigma_f^2 = \sigma_x^2 + \sigma_y^2$	$\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}$
$f = x - y$	$\frac{\partial f}{\partial x} = 1$ and $\frac{\partial f}{\partial y} = -1$	$\sigma_f^2 = \sigma_x^2 + \sigma_y^2$	$\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}$
$f = x\,y$	$\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$	$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$	$\sigma_f = f\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$
$f = x\,/\,y$	$\frac{\partial f}{\partial x} = \frac{1}{y}$ and $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$	$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$	$\sigma_f = f\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$

Example 3

What is the area and associated error for the rectangle if



$$x = 1.0 \pm 0.1\,\mathrm{m}$$

$$y = 2.0 \pm 0.2\,\mathrm{m}$$

Answer: $A = 2.00 \pm 0.28\,\mathrm{m}^2$

Example 4

Suppose we wish to calculate the average speed (displacement/time) of an object. Assume the displacement is measured as $x = 22.2 \pm 0.5\,\mathrm{cm}$ during the time interval $t = 9.0 \pm 0.1\,\mathrm{s}$. What is the speed of the object and its uncertainty?

Answer: $v = 2.467(62)\ \mathrm{cm/s}$

Combining Results of Independent Experiments

Consider we have n independent experiments with results x_i and errors σ_i ($i = 1, \dots, n$).

We can combine the results from each experiment to form a more accurate result. For this, a weighted sum is performed where experiments with smaller errors contribute more to the combined result.

The statistically correct way to combine independent results is as follows:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i / \sigma_i^2}{\sum_{i=1}^{n} 1/\sigma_i^2} \qquad \bar{\sigma} = \frac{1}{\sqrt{\sum_{i=1}^{n} 1/\sigma_i^2}} \qquad (6)$$

Example 5

Given three independent the measurements for the gravitational acceleration data:

$$9.77 \pm 0.14\,, \quad 9.82 \pm 0.10\,, \quad 9.86 \pm 0.20 \text{ m/s}^2$$

Find the combined result of *g*.

Answer: $g = 9.811(75) \text{ m/s}^2$

Statistical & Systematical Errors

Quote statistical and systematic errors separately if both of them are known.

For example:

$$9.81 \pm 0.04\,(\text{stat.}) \pm 0.10\ (\text{syst.})\ \text{m/s}^2$$

Total Error: $\sqrt{(0.04)^2 + (0.10)^2} = 0.11\,\text{m/s}^2$

$$9.81 \pm 0.11\,\text{m/s}^2$$

BİNGÜL, Ahmet
Ph.D. in Engineering Physics
Supervisor: Assoc. Prof. Dr. Ayda BEDDALL
Co-supervisor: Assist. Prof. Dr. Andrew BEDDALL
April 2007, 143 pages

The inclusive production rate of the charged vector meson $\rho^{\pm}(770)$ in hadronic Z decays is measured with the ALEPH detector at the LEP collider. A total of 3,239,746 hadronic events are selected from data recorded by ALEPH from the 1991 to 1995 running periods. Decays of $\rho^{\pm} \to \pi^0 + \pi^{\pm}$ are reconstructed for $x_E > 0.05$ and $x_p > 0.05$ where $x_E = E_\rho / E_{beam}$ and $x_p = p_\rho / p_{beam}$. The $\rho^{\pm}$ multiplicity per hadronic event is evaluated to be:

$$\langle N_{\rho^{\pm}} \rangle = 2.59 \pm 0.03 \pm 0.15$$

where the first error is statistical and the second systematic.

How to compare Experiment and Prediction

In most lab experiments, the result you obtain can be compared directly with a theoretical expectation or a ‘book value’.

Suppose that the accepted value is 0.0 (zero).

* If your measurement is 1.3 ± 1.4

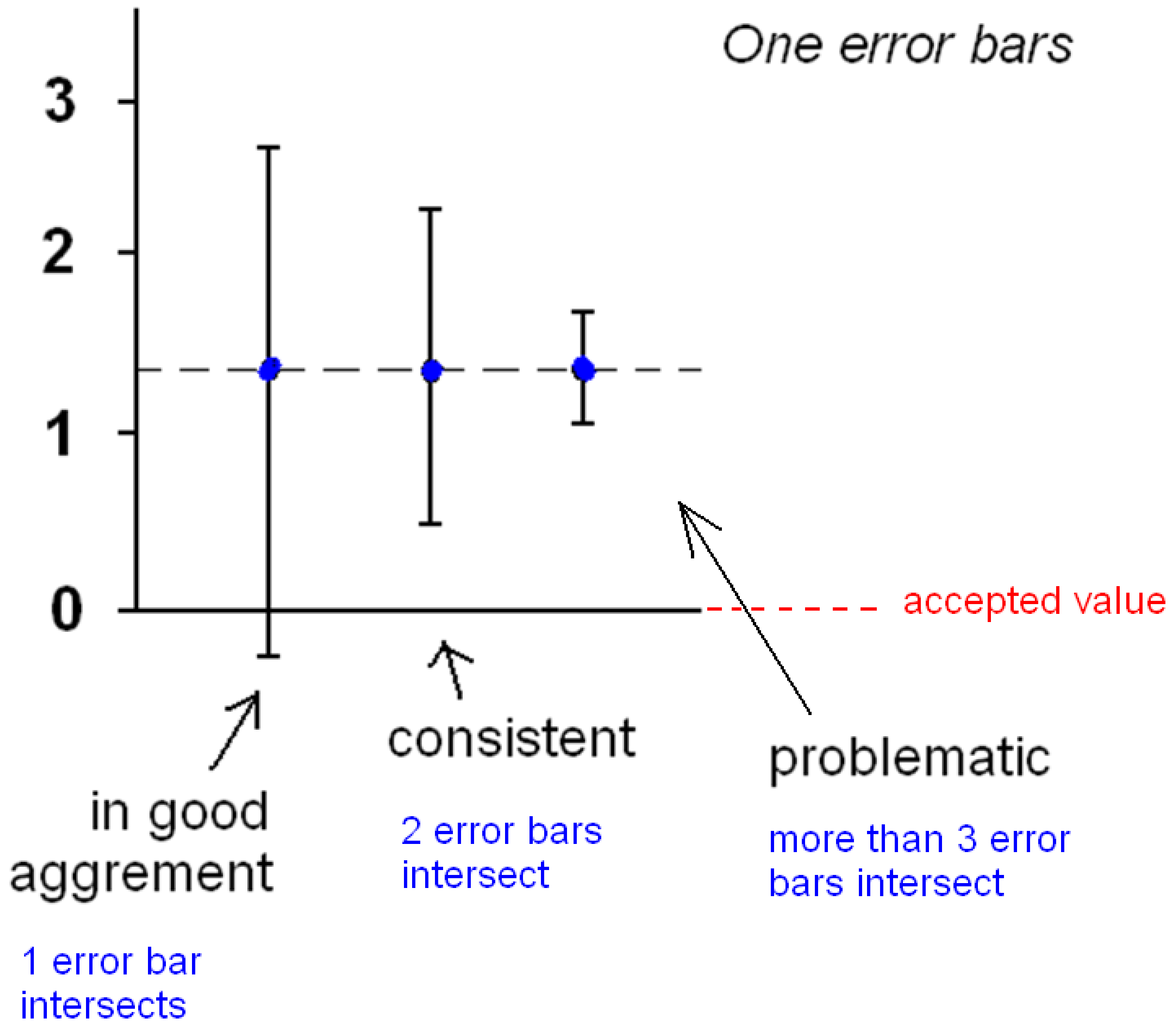
  the results are in good aggrement since 1.3/1.4 = 0.93 < 1

* If your measurement is 1.3 ± 0.8

  the results are consistent since 1.3/0.8 = 1.63 < 2

* If your measurement is 1.3 ± 0.3

  there is a problem! since 1.3/0.3 = 4.33 > 3


One error bars
3
2
1
0
accepted value
in good aggrement
consistent
problematic
1 error bar intersects
2 error bars intersect
more than 3 error bars intersect

Questions

1. By measuring yourself with 10 different rulers, you obtain the estimates of your height.

(a) Which of these is the best estimate of your height?

(b) Use Eqn. (1), (2) and (3), without considering errors, calculate the mean height and standard error of the data.

165.6 ± 0.3 cm
165.1 ± 0.4 cm
166.4 ± 1.0 cm
166.1 ± 0.8 cm
165.5 ± 0.5 cm
165.5 ± 0.4 cm
165.9 ± 0.6 cm
165.5 ± 0.2 cm
166.0 ± 0.7 cm
164.9 ± 0.4 cm

(c) Use Eqn (6), calculate the height and its standard deviation by combining these 10 measurements.

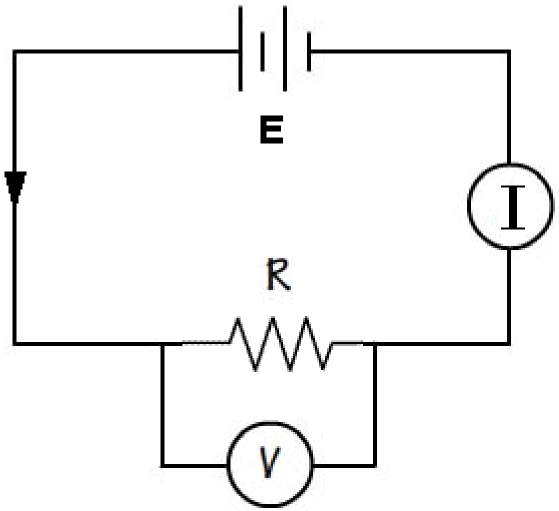
2. Determine the distance between the points A and B.

$$A(0.0 \pm 0.2\ \text{cm},\ 0.0 \pm 0.3\ \text{cm})$$
$$B(3.0 \pm 0.3\ \text{cm},\ 4.0 \pm 0.2\ \text{cm})$$

3. Suppose that $x = 2.0 \pm 0.2$ $y = 3.0 \pm 0.6$ $z = 4.52 \pm 0.02$
Find w = x + y - z and its uncertainty.

4. A resistance *R* is connected in a circuit as shown in figure. Calculate power in the resistor and its uncertainty, if

$$R = 10\,\Omega \pm 1\%$$
$$V = 100\ \text{V} \pm 1\%$$
$$I = 10\ \text{A} \pm 1\%$$



5. The resistance of a certain size of copper wire is given as

$$R = R_0[1 + \alpha(T - 20)]$$

where

R_0 is the resistance at 20 ^{o}C,

alpha is the temperature coefficient of resistance and

T is the resistance of the wire.

Calculate the resistance (R) of the wire and its uncertainty for:

$$R_0 = 6.00 \ \pm 0.02 \ \Omega$$

$$\alpha = (4.00 \ \pm 0.04) \times 10^{-3} \ {}^{o}C^{-1}$$

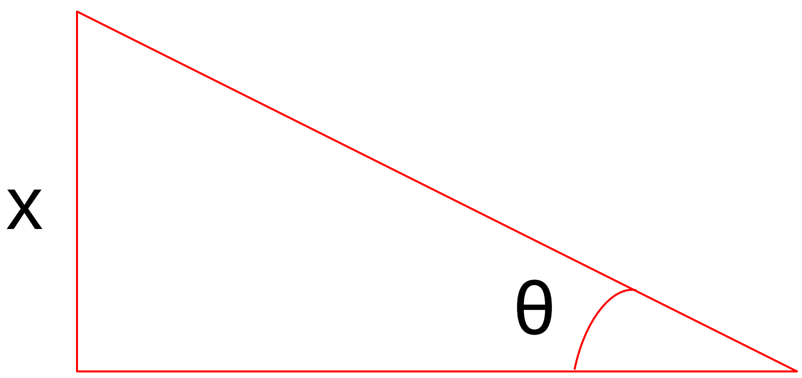
$$T = 30 \ \pm \ 1 \ {}^{o}C$$

6. Two resistors R_1 and R_2 are to be connected in series and parallel. The values of the resistances are

$$R_1 = 100.0 \pm 0.3\,\Omega, \quad R_2 = 50.0 \pm 0.1\,\Omega$$

Calculate the value and uncertainty in the combined resistance for both series and parallel arrangements.

7. Figure shows a right angle triangle.
Find the area of the triangle and its uncertainty for



$$\theta = 37.0^{\circ} \pm 0.4^{\circ}$$

$$x = 3.0 \pm 0.2\,\mathrm{m}$$

8. Refractive index (n) of a glass is calculated by using Snell's law:

$$n = n_{\mathrm{AIR}} \frac{\sin \theta_1}{\sin \theta_2}$$

where measured values and their estimated errors are:

$$n_{\mathrm{AIR}} = 1 \qquad \theta_1 = 61 \pm 1^{\mathrm{o}} \qquad \theta_2 = 36 \pm 1^{\mathrm{o}}$$

Calculate the refractive index of the glass and its uncertainty.

9. Four independent measurements for the speed of light (c) are given by:

$$c_1 = 299788 \pm 30 \text{ km/sec}$$

$$c_2 = 299796 \pm 4 \text{ km/sec}$$

$$c_3 = 299792 \pm 2 \text{ km/sec}$$

$$c_4 = 299792.5 \pm 0.1 \text{ km/sec}$$

Calculate the combined value of the speed of light and combined uncertainty.

10. Compare the two independent measurements for the speed of light (c).

$$c_1 = 299796 \pm 4 \text{ km/sec}$$

$$c_2 = 299792 \pm 2 \text{ km/sec}$$

Compute how many error bars first measurement differ from the second measurement.

Answer: 0.89

$$\Delta c = | c_1 - c_2 | = 4.00 \text{ km/sec}$$

$$\sigma_c = \sqrt{\sigma_1^2 + \sigma_2^2} = 4.47 \text{ km/sec}$$

$$\frac{\Delta c}{\sigma_c} = \frac{4.00}{4.47} = 0.89$$

References

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