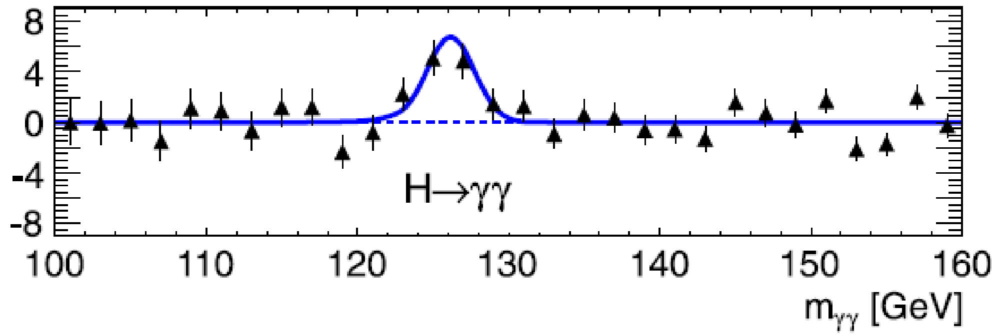
EP122
Measurement Techniques and Calibration

Topic 3

Introductory Probability



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Content

PART I

BASIC SET THEORY

Set Theory

In mathematics, a well defined collection of objects is called the **set**.

Examples:

$A = \{1,2,3,4\}$	=> finite set
$M = \{\text{apple, banana, orange}\}$	=> finite set
$R = \{x \mid x \text{ is a river on Earth}\}$	=> finite set
$N = \{0,1, 2, 3, 4, ...\}$	=> infinite set
$P = \{2, 4, 8, ...\}$	=> infinite set
$K = \{x \mid 2<x<5, x \text{ is a real}\}$	=> infinite set

Notation:

$p \in A$ — *p* is elements of *A*

$A \subset B$ — *A* is subset of *B*

U — Universal set

ϕ — Empty set

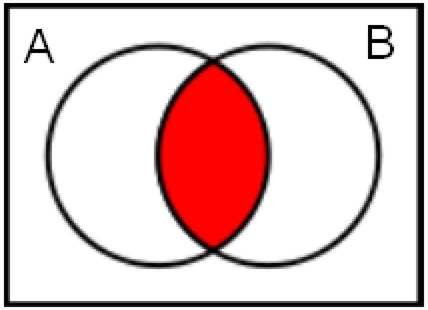
For any set A

$$\phi \subset A \subset U$$

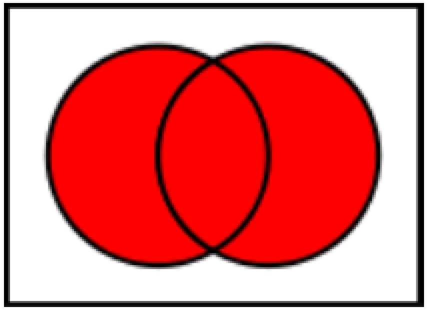
Set Operations:

Venn Diagram

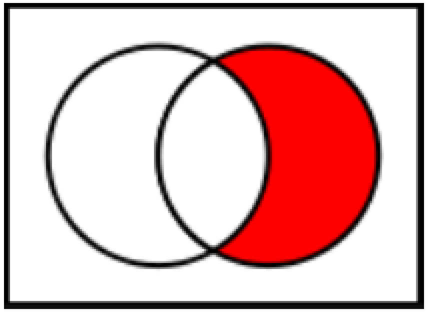
Intersection $A \cap B$



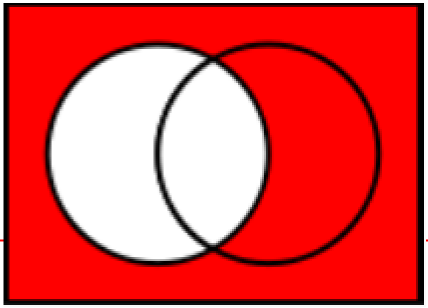
Union $A \cup B$



Difference $B - A$



Complement A^C



Example

Let

A ={1, 2, 3, 4}, B = {3, 4, 5, 6} and U={1, 2, 3, 4, 5, 6, 7,8 …}

A U B = {1, 2, 3, 4, 5, 6}

A n B = {3, 4}

A \ B = {1, 2}

A^c = {5, 6, 7, 8, ...}

PART II

BASIC PROBABILITY

Probability (=Olasılık)

Historically, the probability theory began with study of games of chance, such as roulette and cards.

The probability is the study of random or non-deterministic experiments

If a coin is tossed in the air, then

it is certain that the coin will come down

but

it is not certain that a head will appear.

Relative Frequency (=Göreli Sıklık)

Suppose we repeat an experiment of tossing a die.

Let

s be the number of times a “six” appears

n be the number of tosses

Then the ratio s/n becomes <u>stable</u> in the long run:

$$f = \frac{s}{n}$$

f approaches a limit as n -> ∞

This stability is the basis of probability theory!

Example

Here is the result obtained from a computer simulation for tossing of a coin and observing frequency of head!

HEAD

TAIL

n	s	f = s/n
10	4	0.4000000
100	41	0.4100000
1,000	476	0.4760000
10,000	5059	0.5059000
100,000	49942	0.4994200
1,000,000	500351	0.5003510
10,000,000	4998906	0.4998906
100,000,000	50006417	0.5000641
1,000,000,000	500000839	0.5000084

The result approaches a limit as $n \to \infty$

Example

Here is the result obtained from a computer simulation for tossing of a die and observing frequency of six!

n	s	f = s/n
10	3	0.3000000
100	19	0.1900000
1,000	186	0.1860000
10,000	1659	0.1659000
100,000	16748	0.1674800
1,000,000	166705	0.1667050
10,000,000	1667210	0.1667210
100,000,000	16666290	0.1666629
1,000,000,000	166666653	0.1666666

The result approaches a limit as $n \to \infty$

Probability Theory

The probability *p* of an event *A* is defined as follows:

If *A* occurs in *s* ways out of a total *n* equally likely ways then

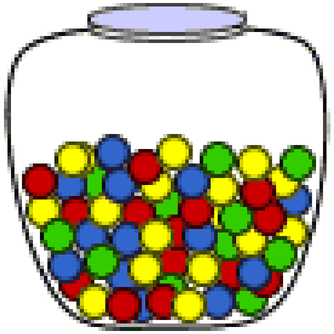
$$p = P(A) = \frac{s}{n}$$

* Tossing a coin: Head occurs 1 way out of 2 => p = 1/2
* Tossing a die: Six occurs 1 way out of 6 => p = 1/6
* Tossing a die: Even number occurs 3 ways out of 6 => p = 3/6

Probability is the measure of how likely an event is

Example

A glass jar contains 6 red, 5 green, 8 blue and 3 yellow marbles. If a single marble is chosen at random from the jar, what is the probability of choosing a red marble? a green marble? a blue marble? a yellow marble?



Outcomes: The possible outcomes of this experiment are red, green, blue and yellow.

Probabilities:

$$P(\text{red}) = \frac{\text{\# of ways to choose red}}{\text{total \# of marbles}} = \frac{6}{22} = \frac{3}{11}$$

$$P(\text{green}) = \frac{\text{\# of ways to choose green}}{\text{total \# of marbles}} = \frac{5}{22}$$

$$P(\text{blue}) = \frac{\text{\# of ways to choose blue}}{\text{total \# of marbles}} = \frac{8}{22} = \frac{4}{11}$$

$$P(\text{yellow}) = \frac{\text{\# of ways to choose yellow}}{\text{total \# of marbles}} = \frac{3}{22}$$

Sample Space (=Örneklem Uzayı)

The set *S* of all possible outcomes of some given experiment is called "sample space"

*** Tossing a coin:** S = {H, T}

*** Tossing two coins:** S = {HH, HT, TH, TT}

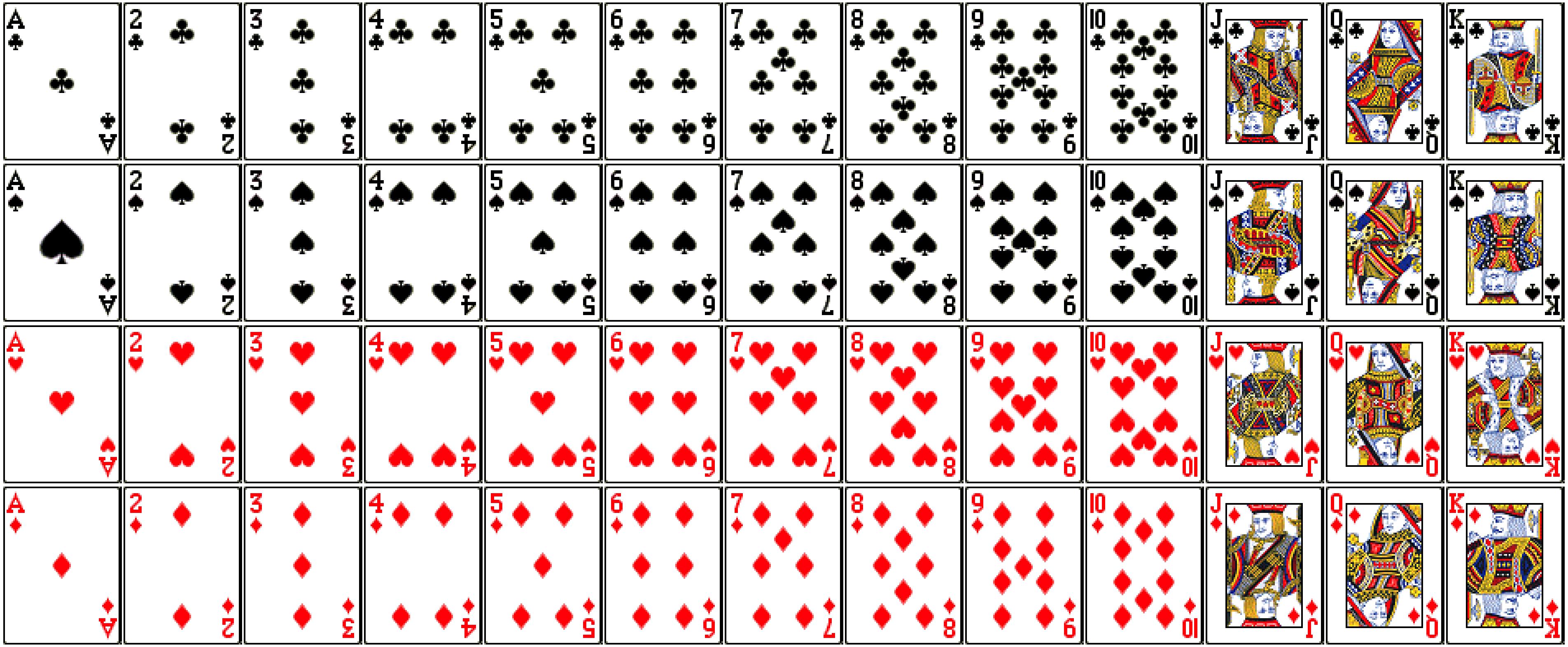
*** Tossing a die**:

S = {1, 2, 3, 4, 5, 6}

*** For two dice the outcomes are**

S = {11,12,13,14,15,16,
21,22,23,24,25,26,
31,32,33,34,35,36,
41,42,43,44,45,46,
51,52,53,54,55,56,
61,62,63,64,65,66}

Selecting a card randomly from a shuffle pack of playnig cards, the possible outcomes are:



Axioms of Probability

Let *S* be sample space and *A* and *B* are two events.

A1. $0 \leq P(A) \leq 1$

A2. $P(S) = 1$

A3. If *A* and *B* are mutually exclusive events (ayrık olaylar)

$$P(A \cup B) = P(A) + P(B)$$

Theorems of Probability

T1. $P(\phi) = 0$ (probability of impossible event is zero)

T2. If A^c is the complement of *A*, then $P(A^c) = 1 - P(A)$

T3. If *A* and *B* are <u>any</u> two events:

$$P(A - B) = P(A) - P(A \cap B)$$

T4. If *A* and *B* are <u>any</u> two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) - P(A \cap B \cap C)$$

Example

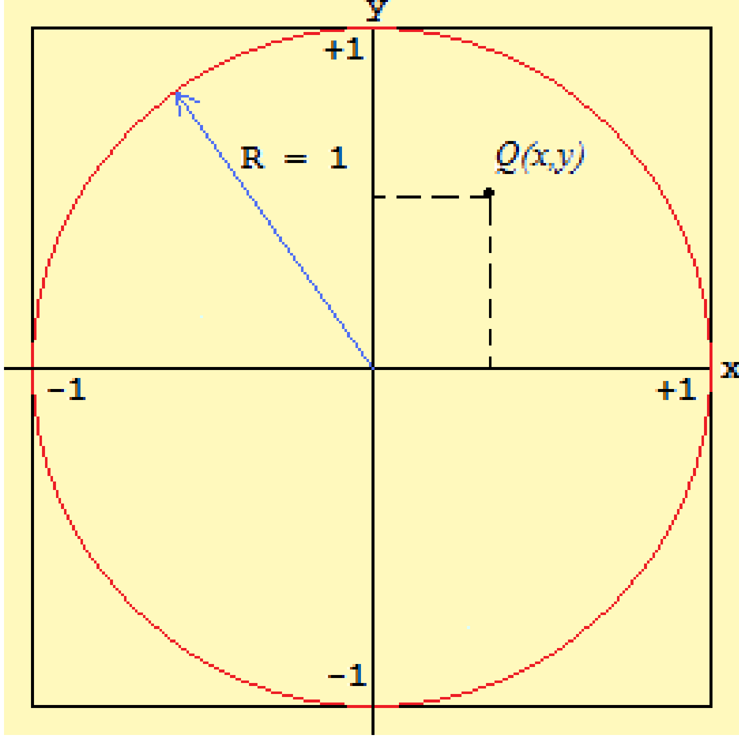
(a) What is the sample space for choosing 1 letter at random from the word DIVIDE?

(b) What is the probability of selecting the letter V from the word DIVIDE?

Example

A point Q is selected randomly in a square whose side is 2 cm. A circle is drawn tangent to the edges of the square.

Find the probability of the point being inside the circle.



PART III

RANDOM VARIABLES

Random Variable (=Raslantı Değişkeni)

- A Random Variable (RV) is a set of values that occur randomly and have associated probabilities.
- RV can be discrete (kesikli) or continues (sürekli).
- Probability mass function (pmf) describes the distribution of the discrete probabilities
- Probability distribution function (pdf) describes the distribution of the continues probabilities.

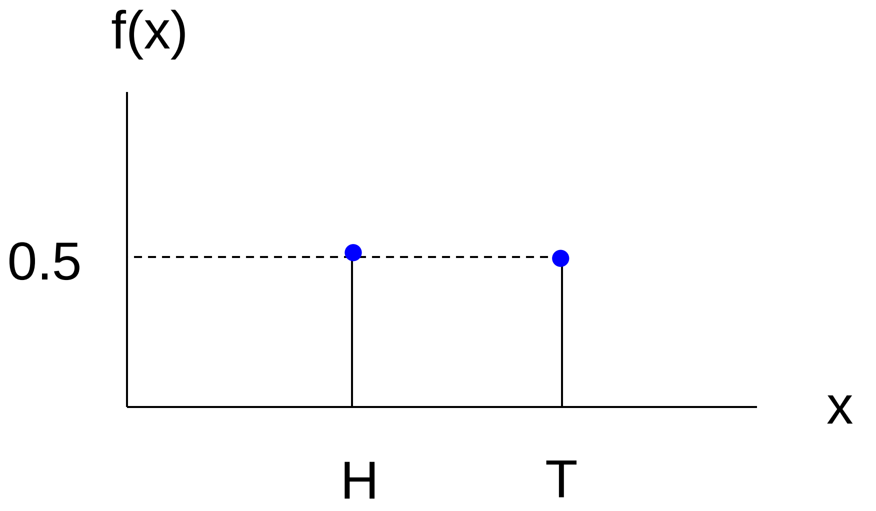
Discrete RV Examples:

pmf

Tossing a coin:

X = {H,T}

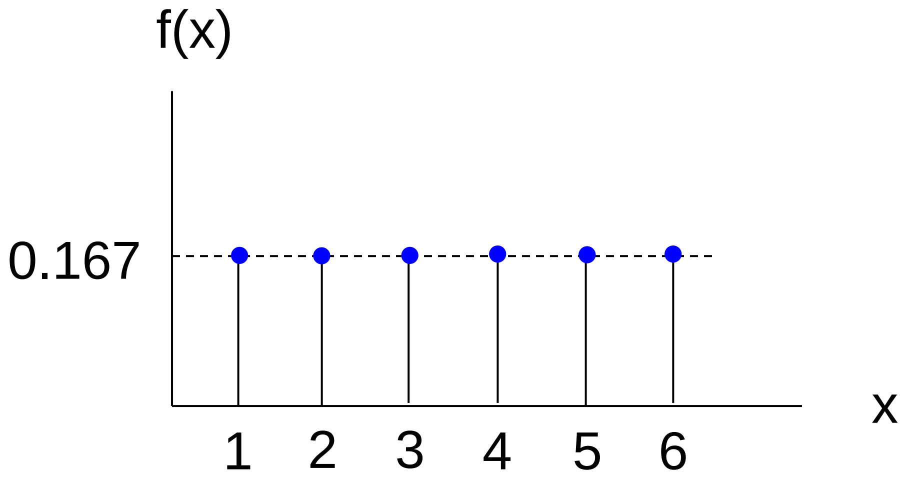
f(x) = {1/2, 1/2}



Tossing a die:

X = {1,2,3,4,5,6}

f(x) = {1/6,1/6,1/6,1/6, 1/6, 1/6}

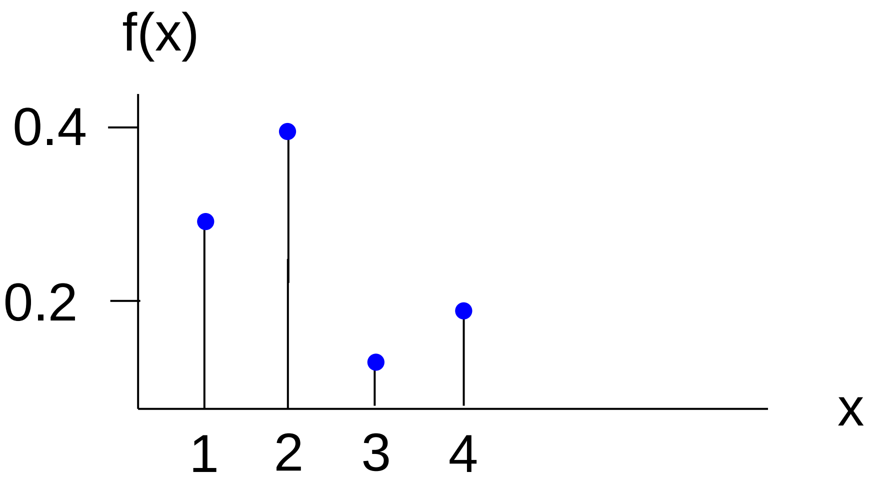


Discrete RV Examples:

X = {1,2,3,4}

f(x) = {0.3, 0.4, 0.1, 0.2}

pmf



Continues RV Examples:

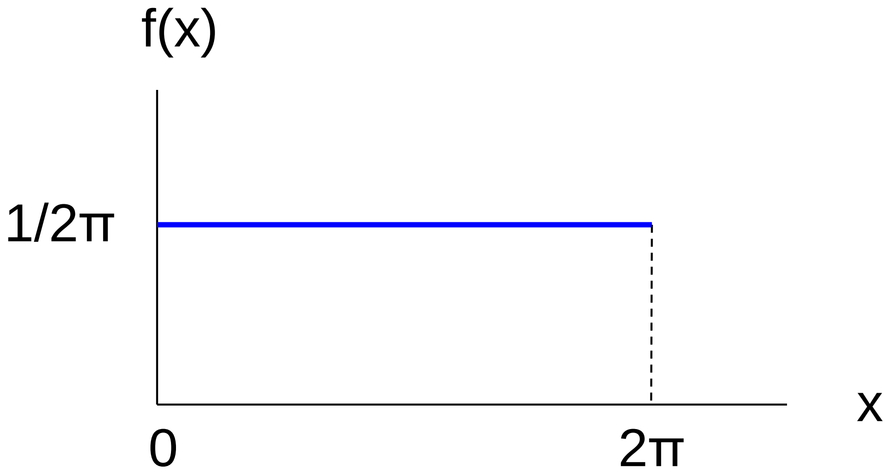
pdf

Random angle

in the range [0, 2π]

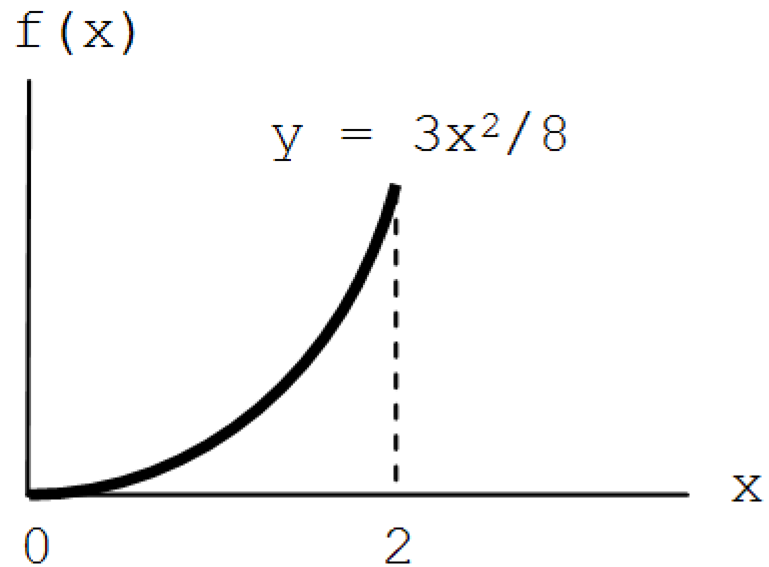
X = [0, 2π]

f(x) = 1/2π



X = [0, 2]

$f(x) = 3x^2/8$



Properties of Random Variables

Discreate RV

$$f(x_i) \geq 0$$

$$\sum_i f(x_i) = 1$$

$$\sum_{i=a}^{b} f(x_i) = P(a \leq x \leq b)$$

Continues RV

$$f(x) \geq 0$$

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

$$\int_a^b f(x)dx = P(a \leq x \leq b)$$

Expectation Values

Expectation Value or Mean Value of RV is denoted by:

$$E\,[\,X\,] \qquad \text{or} \qquad < X > \qquad \text{or} \qquad \bar{x}$$

and defined by:

Discrete RV: $$E[X] = \sum_i x_i \; f(x_i)$$

Continues RV: $$E[X] = \int_{-\infty}^{+\infty} x \; f(x) dx$$

Mean of the Squares

It is defined as:

Discrete RV: $$E[X^2] = \sum_i x_i^2 \ f(x_i)$$

Continues RV: $$E[X^2] = \int_{-\infty}^{+\infty} x^2 \ f(x)dx$$

Note that (**RMS**: Root Mean Square)

$$RMS = \sqrt{E[X^2]}$$

Variance

Variance is defined as:

Discrete RV: $$\sigma^2 = \sum_i (x_i - \bar{x})^2 \; f(x_i)$$

Continues RV: $$\sigma^2 = \int_{-\infty}^{+\infty} (x - \bar{x})^2 \; f(x) dx$$

One can also prove that

$$\begin{aligned} \sigma^2 &= E[X^2] - (E[X])^2 \\ &= \langle X^2 \rangle - \langle X \rangle^2 \end{aligned}$$

Standard Deviation

Square root of variance is called **standard deviation**:

$$\sigma = \sqrt{\sigma^2}$$

$$\sigma = \sqrt{E[X^2] - (E[X])^2}$$

$$= \sqrt{<X^2> - <X>^2}$$

There are some applications of this equation in Quantum Mechanics

Example

$$X = \{1, 2, 3, 4, 5, 6\}$$

$$f(x) = \{1/6, 1/6, 1/6, 1/6, 1/6, 1/6\}$$

Find

(a) Expectation value of X, (b) RMS and (c) standard deviation

Example

Suppose a variable X can take the values 1, 2, 3, or 4.

The probabilities associated with each outcome are described by the following table:

Outcome	**1**	**2**	**3**	**4**
Probability	**0.1**	**0.3**	**0.4**	**0.2**

Find

(a) $P(X = 2 \text{ or } X = 3)$

(b) $1 - P(X = 1)$

Example

The following table gives the probability distribution of, X, the number of telephones in a randomly selected home in a certain community.

x:	**0**	**1**	**2**	**3**	**4**
f(x):	**0.021**	**0.412**	**0.283**	**0.188**	**0.096**

One home is selected randomly.

The probability that it will have:

(a) no telephone is 0.021

(b) fewer than two telephones is 0.433

(c) at least three telephones is 0.284

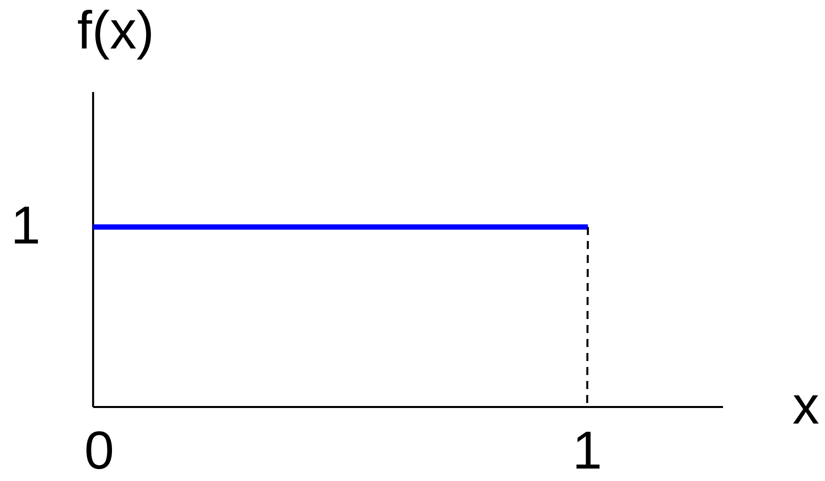
(d) one or two telephones is 0.695

Example

Consider the uniform pdf.

X = [0,1]

f(x) = 1



Find

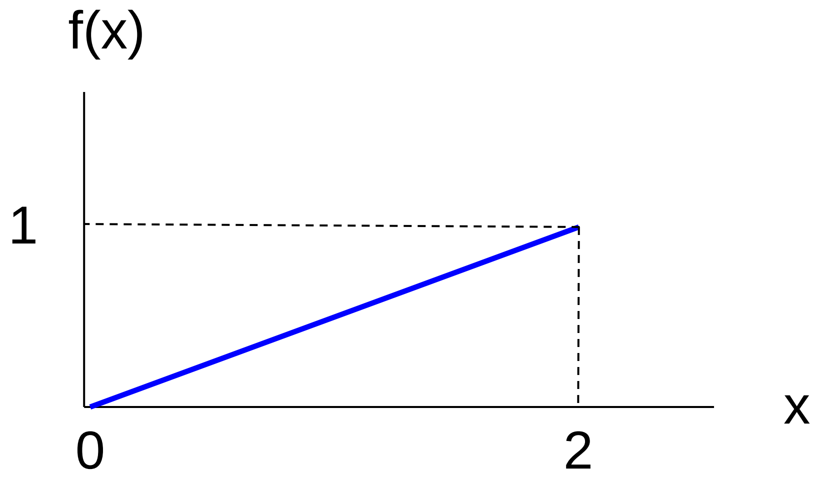
(a) mean, (b) RMS and (c) standard deviation

Example

Consider the triangular pdf.

X = [0,2]

f(x) = kx



Find

(a) the value k (b) mean and (c) standard deviation

PART IV

SPECIAL DISTRIBUTION FUNCTIONS

Binomial Distribution Function

The binomial distribution function specifies the number of times (*k*) that an event occurs in *n* independent trials where *p* is the probability of the event occurring in a single trial.

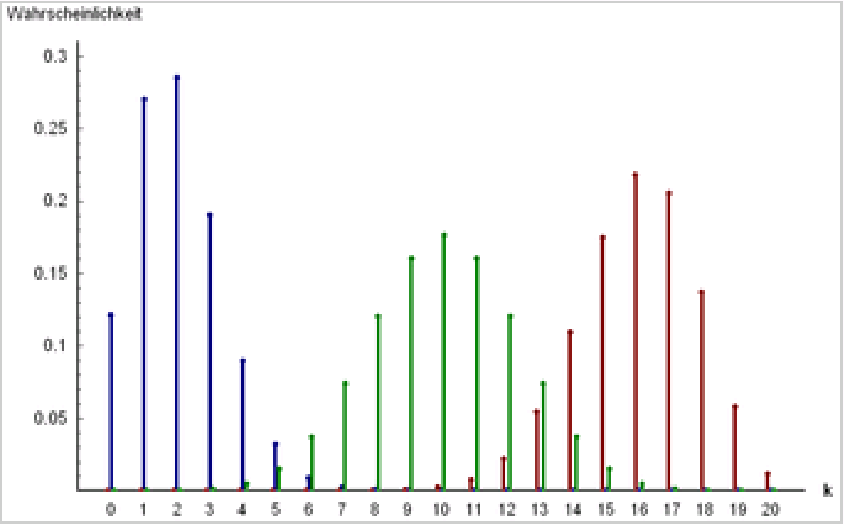
$$p_{binom}(n,k,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\text{mean}: \quad \langle k \rangle = np$$

$$\text{std.dev}: \quad \sigma = \sqrt{np(1-p)}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$



Binomial distribution for $n = 20$
$p = 0.1$ (blue), $p = 0.5$ (green) and $p = 0.8$ (red)

Example: A coin is tossed 3 times.

If you call heads a success, draw pmf for $k = 0, 1, 2, 3$.

Example: A coin is tossed 6 times.

The probability of getting exactly four heads:

$$P = \frac{6!}{4!\,(6-4)!}\left(\frac{1}{2}\right)^{4}\left(1-\frac{1}{2}\right)^{6-4} = 0.234375$$

The probability of getting at least four heads:

$$P = \binom{6}{4}\left(\frac{1}{2}\right)^{4}\left(1-\frac{1}{2}\right)^{6-4} + \binom{6}{5}\left(\frac{1}{2}\right)^{5}\left(1-\frac{1}{2}\right)^{6-5} + \binom{6}{6}\left(\frac{1}{2}\right)^{6}\left(1-\frac{1}{2}\right)^{0}$$

$$= 0.34375$$

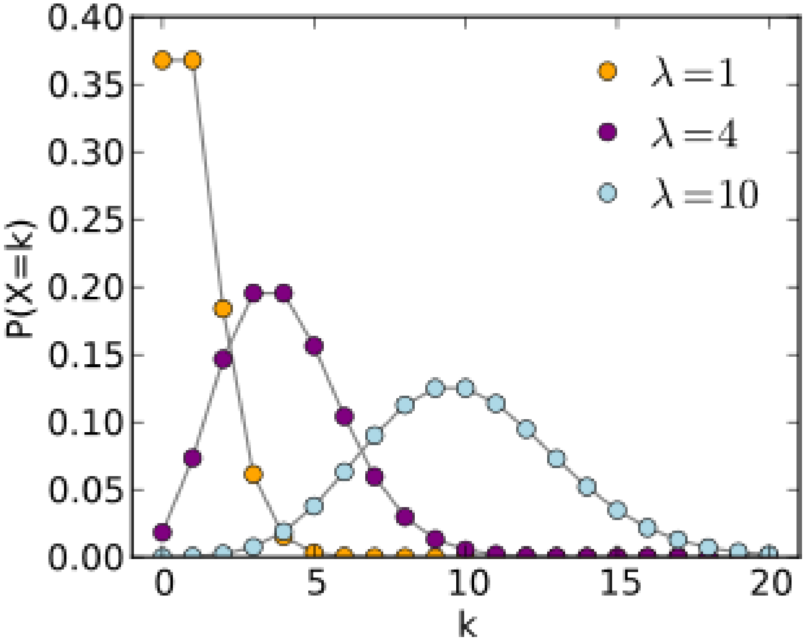
Poisson Distribution Function

In the binomial equation, if the probability *p* is so small then the distribution of events can be approximated by the Poisson distribution.

$$p_{poisson}(k,\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\text{mean}: \quad \lambda = np$$

$$\text{std.dev}: \quad \sigma = \sqrt{\lambda}$$



$\lim_{p \to 0}$ Binomial Distribution = Poisson Distribution

Example: *Birthday problem*

Probability of one person to have birthday in any day is 1/365 = 0.00274. Calculate the probability that 4 people share a birthday in a group of 1000 people.

Mean: $\lambda = 1000*0.00274 = 2.734$

Probability: $p = \exp(-2.734)*2.734^4/4! = 0.151$

Example

Suppose 1% of items made by a factory are defective. Find the probability that 6 defective items in a sample of 300 items.

The Gaussian or Normal Distribution Function

In Statistics, if the number of events is very large (n>20), then the Gaussian (normal) distribution function may be used to describe nearly all events.

The Gaussian distribution is a continuous Random Variable of the form:

$$p_{gauss}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\text{mean}: \quad \mu$$

$$\text{std.dev}: \quad \sigma$$

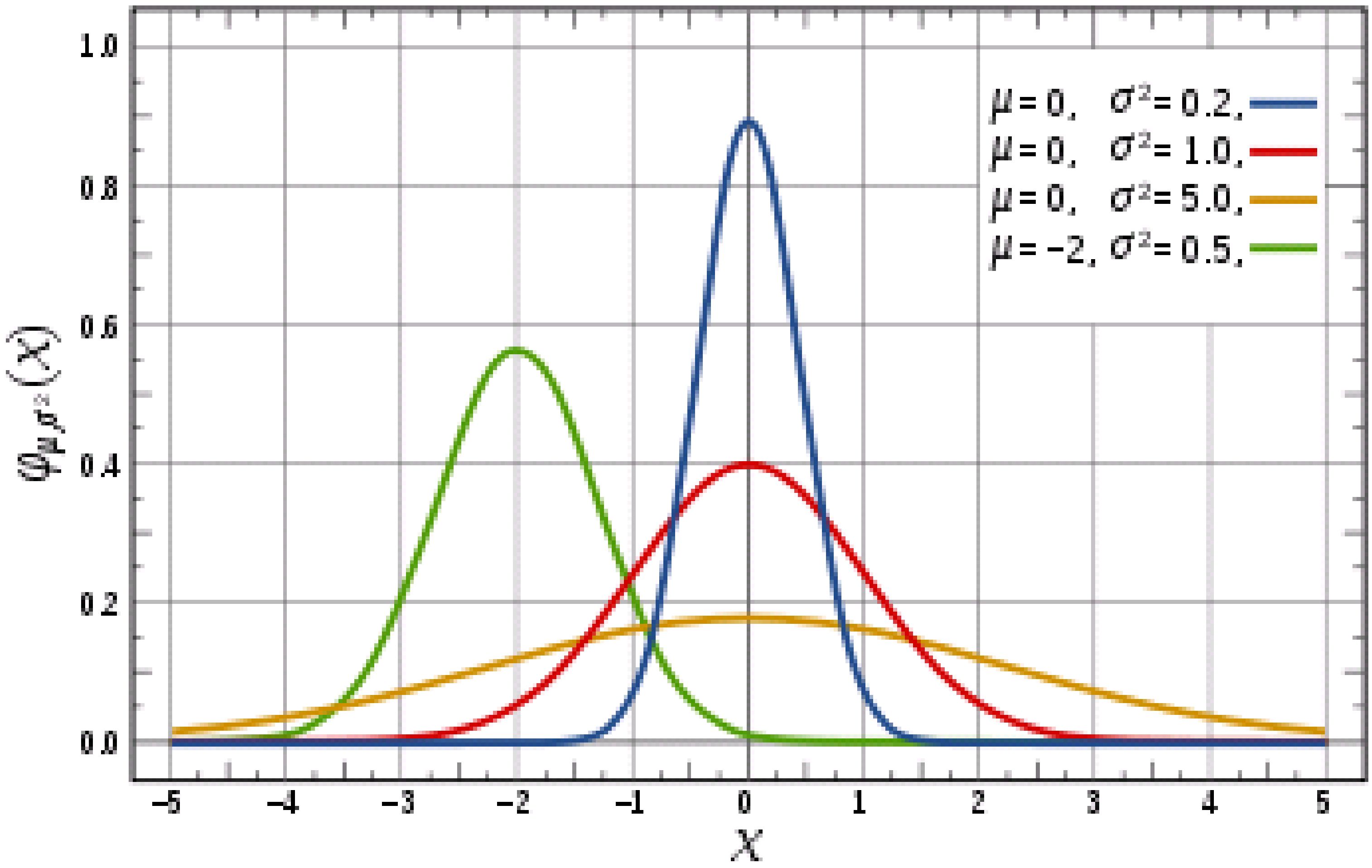
$$p_{gauss}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

μ = mean

σ = standart deviation

π = 3.141593

e = 2.718281



Properties of Gaussian Function

$$p_{gauss}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$p(x) \geq 0$$

$$\int_{-\infty}^{+\infty} p(x)dx = 1$$

$$E[X] = \int_{-\infty}^{+\infty} xp(x)dx = \mu$$

$$\int_{a}^{b} p(x)dx = P(a \leq x \leq b)$$

$$\int_{-\infty}^{+\infty} (x-\mu)^2 p(x)dx = \sigma^2$$

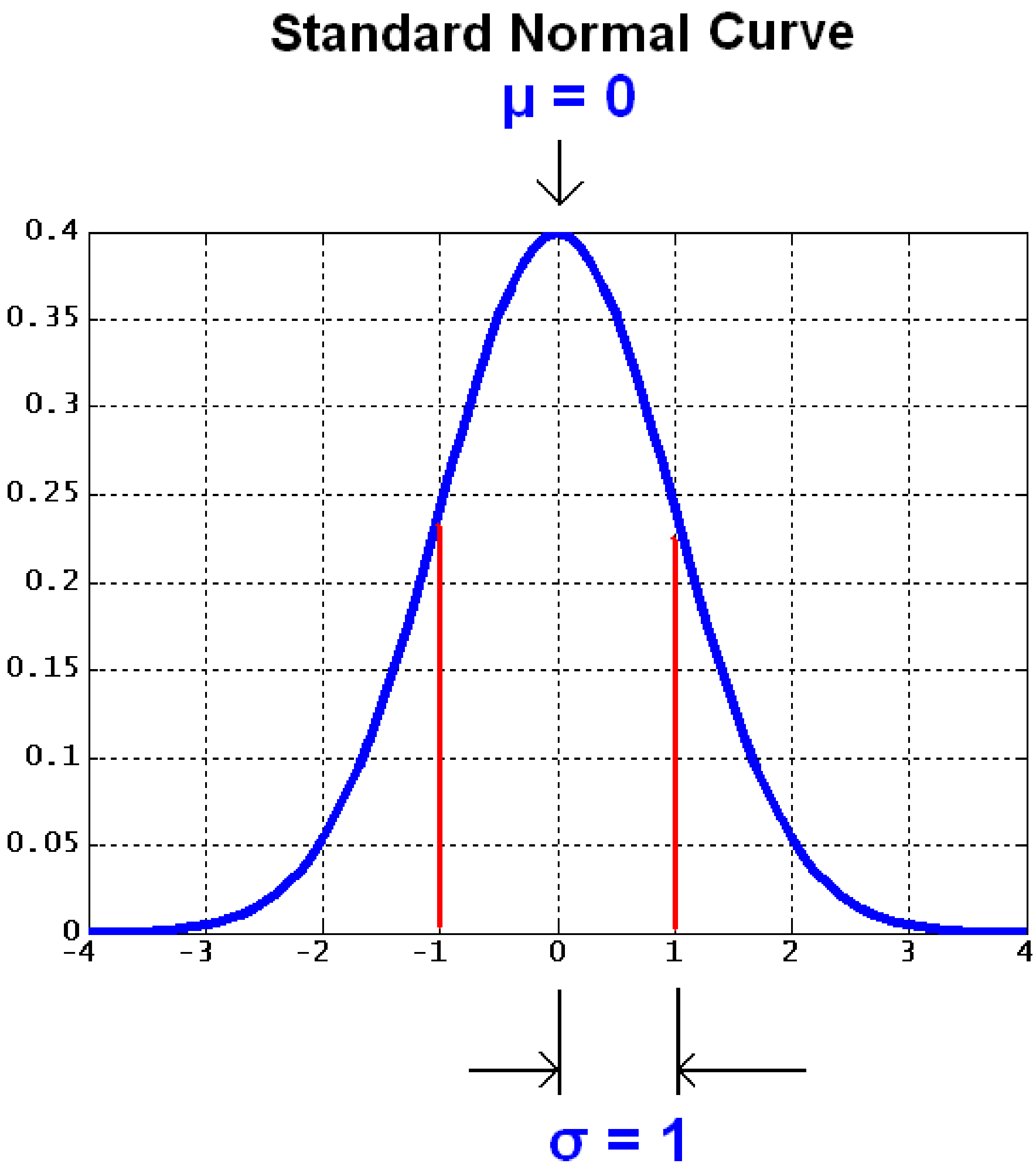
Standard Normal Curve

The normal distribution function for

$\mu = 0$ and $\sigma = 1$

is called the **standard normal distribution function**.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \approx 0.4 \exp(-x^2/2)$$

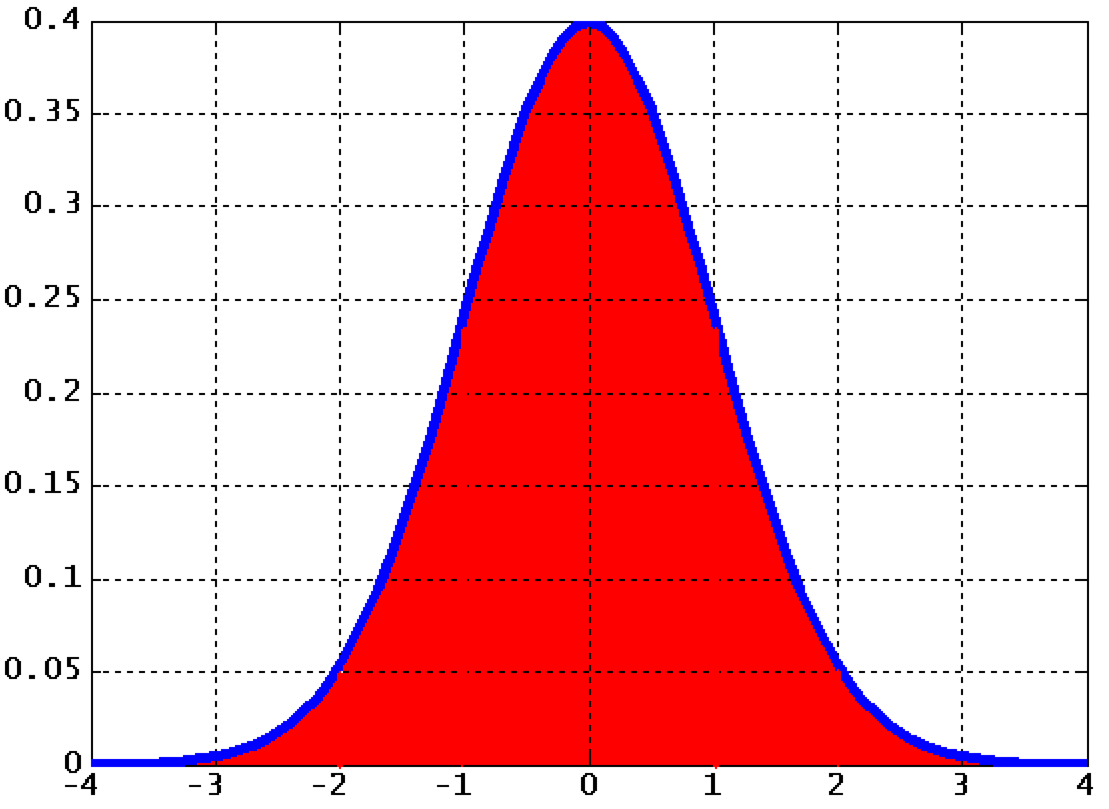

Standard Normal Curve
μ = 0
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-4
-3
-2
-1
0
1
2
3
4
σ = 1

Area Under the Curve

Total area under the standard normal curve is 1.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

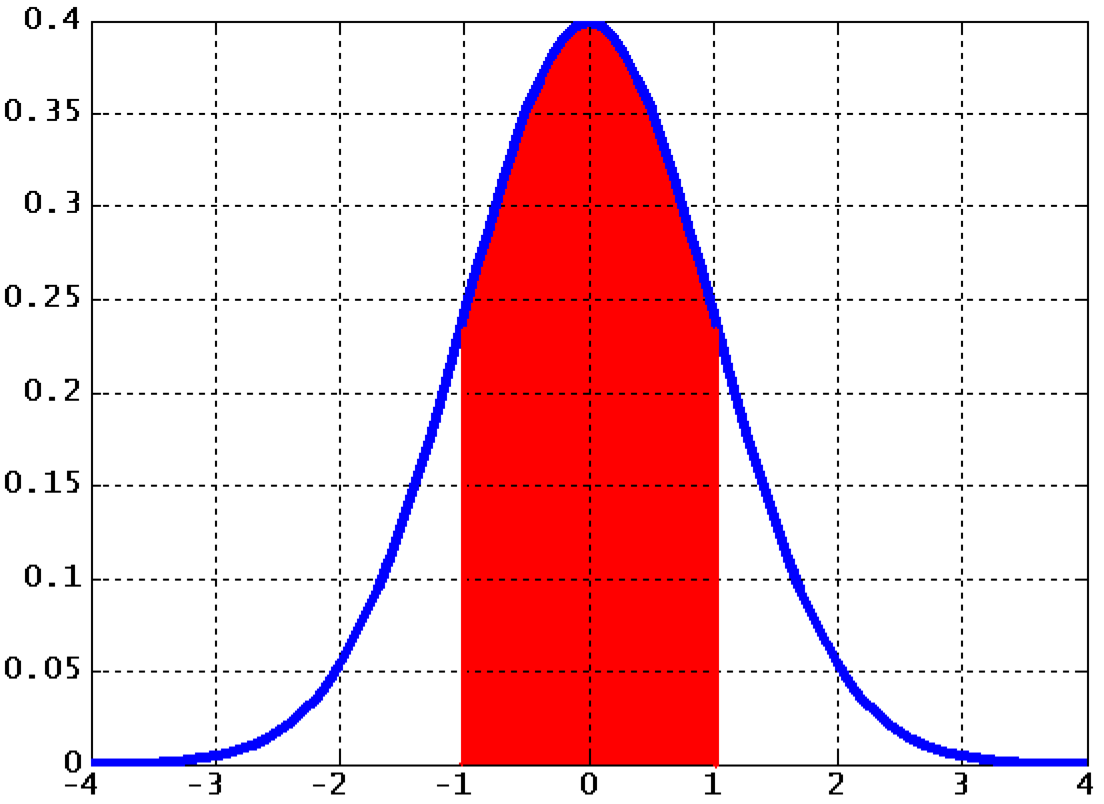


Area under the standard normal curve between [-1, 1] is:

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0.6827$$

This corresponds

+- 1 sigma

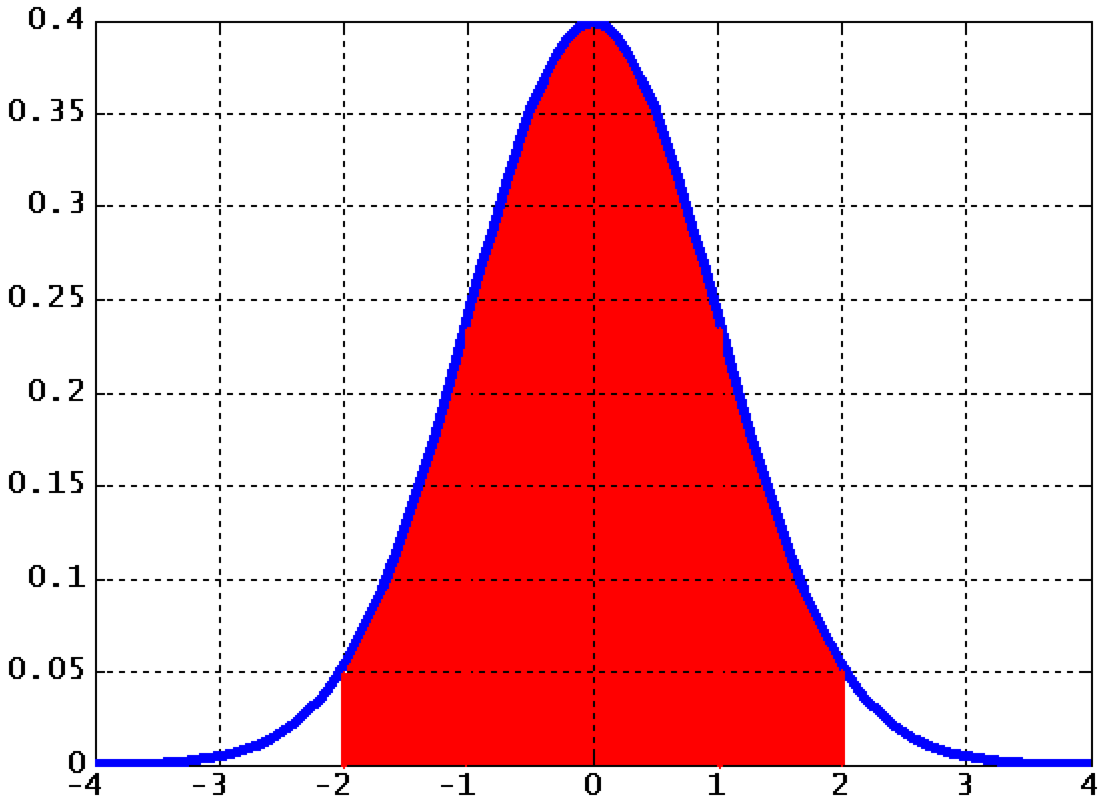


Area under the standard normal curve between [-2, 2] is:

$$\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0.9545$$

This corresponds

+- 2 sigma

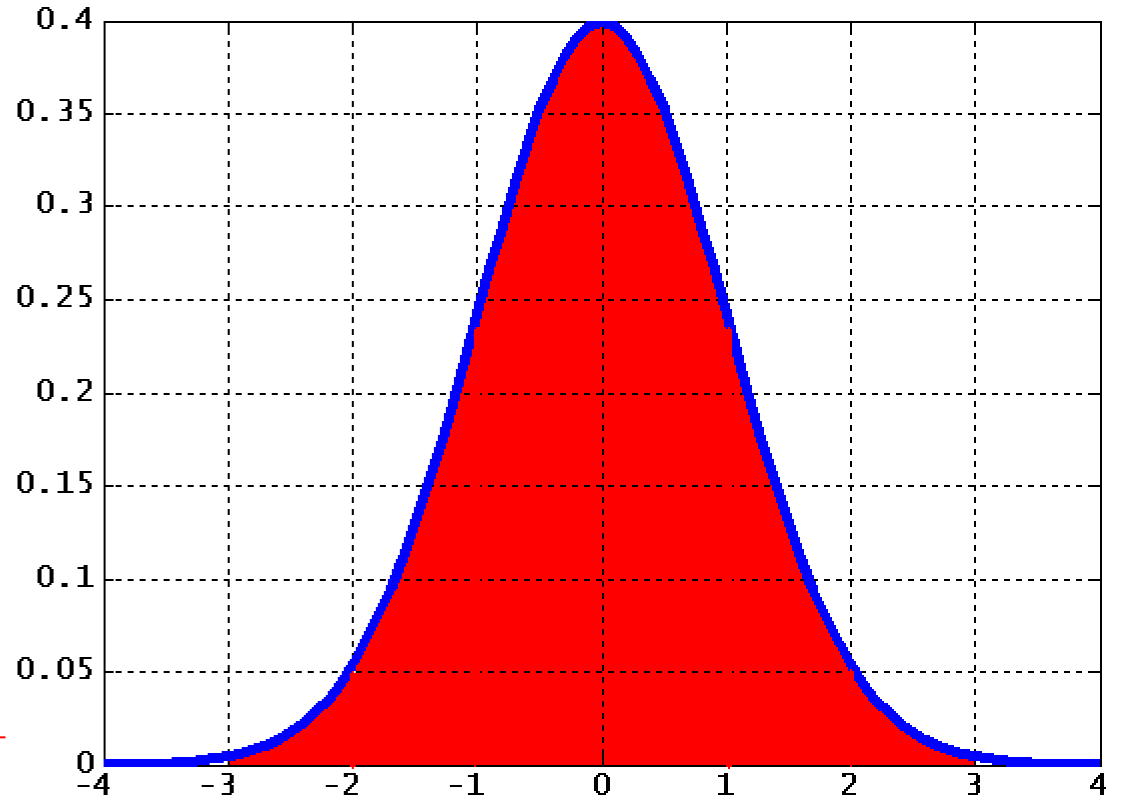


Area under the standard normal curve between [-3, 3] is:

$$\int_{-3}^{3} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0.9973$$

This corresponds

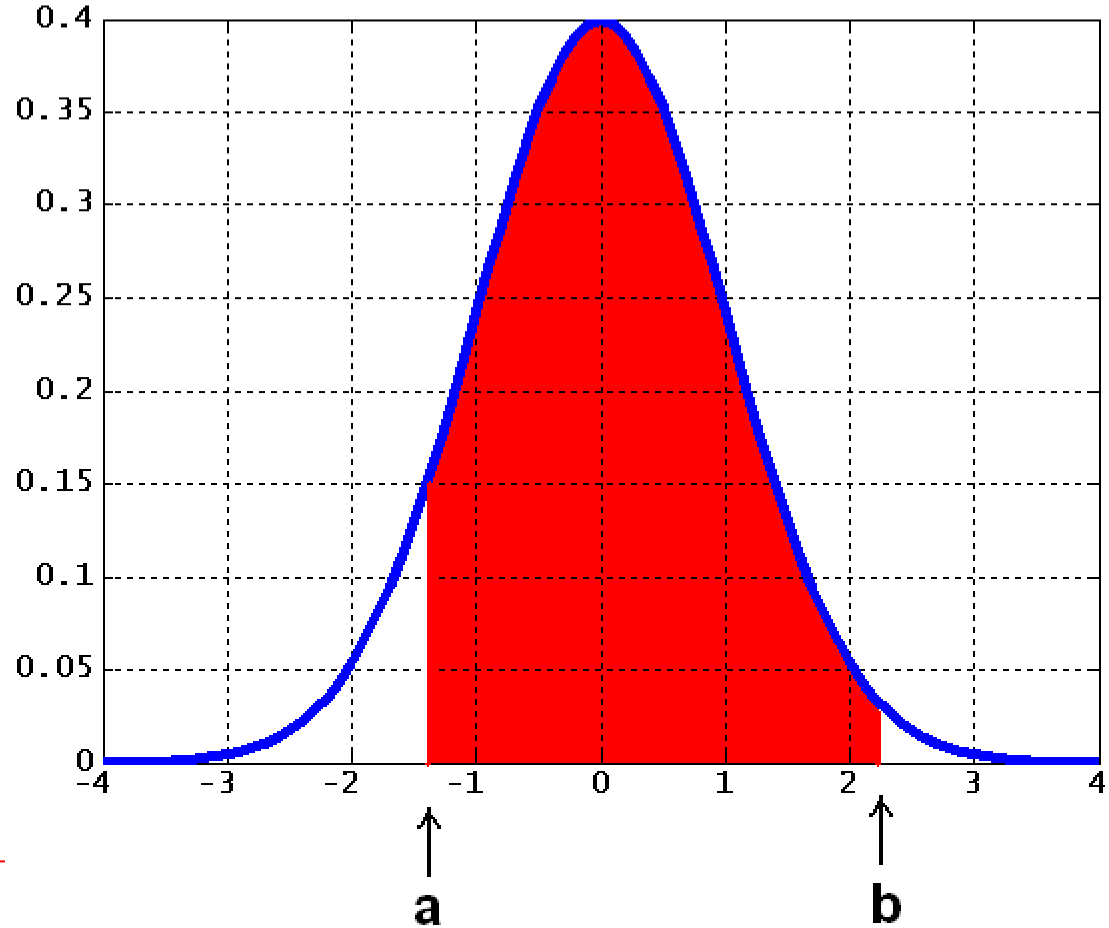
+- 3 sigma

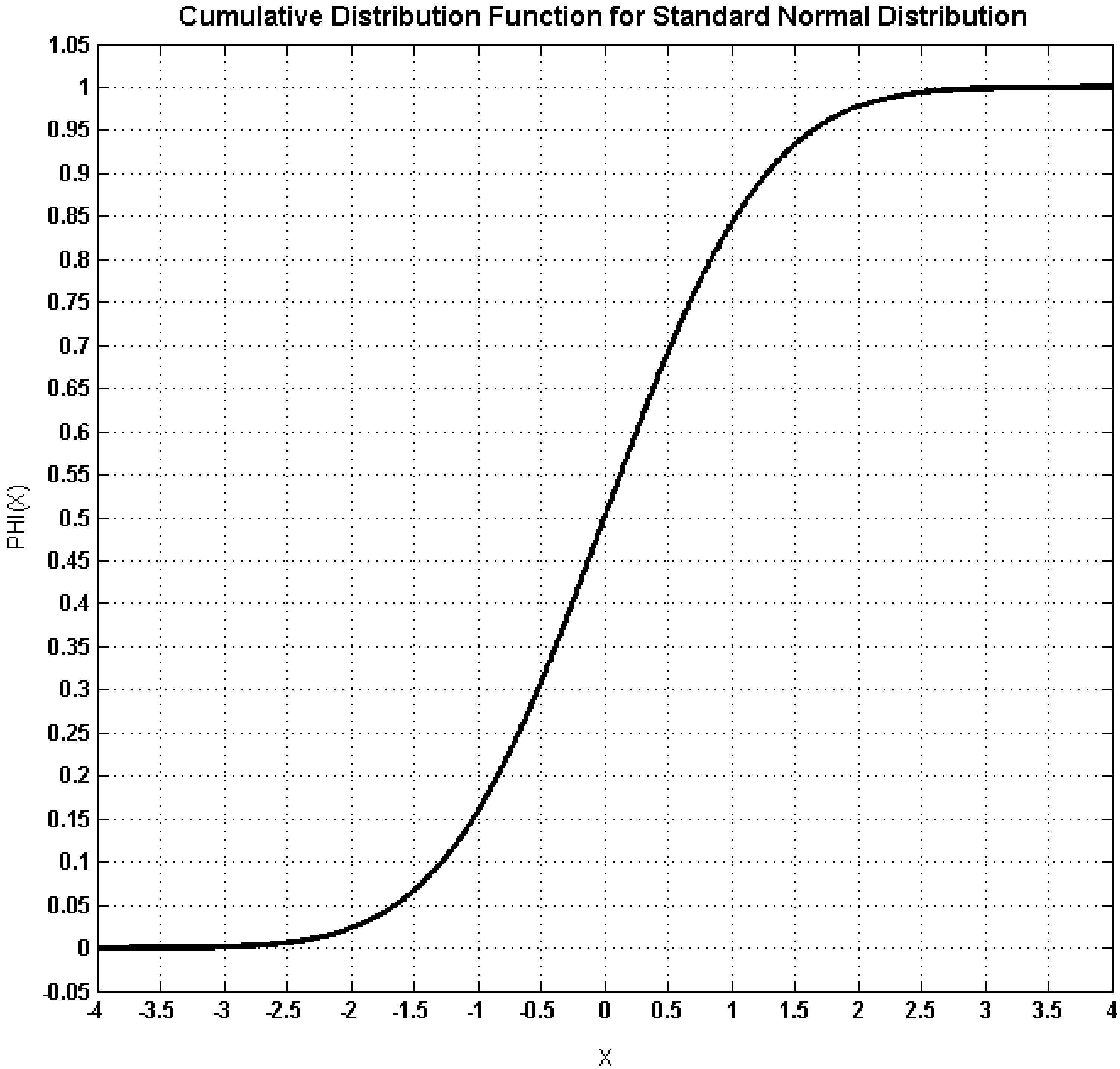


Area under the standard normal curve between [a, b] is:

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \Phi(b) - \Phi(a)$$

The values of the function phi(x) can be taken from a table or from the figure on next page.




Cumulative Distribution Function for Standard Normal Distribution
PHI(X)
X
1.05
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4

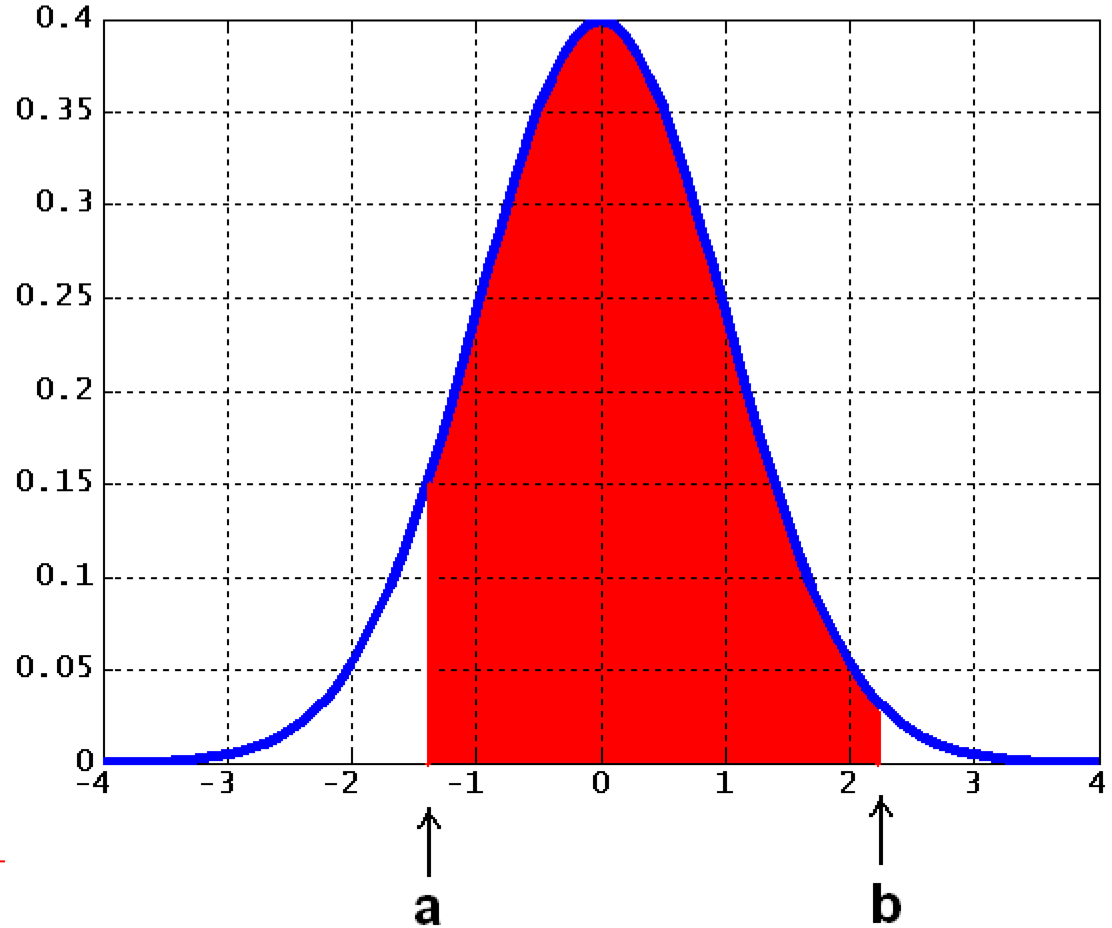
Example:

$$\frac{1}{\sqrt{2\pi}}\int_{-1.2}^{2.3} e^{-x^2/2}dx = \Phi(2.3) - \Phi(-1.2) = 0.99 - 0.12 = 0.87$$

From CDF figure (previous page)

$$\Phi(2.3) \approx 0.99$$

$$\Phi(-1.2) \approx 0.12$$



Alternatively one can use Octave's **`normcdf(x)`** function:

$$\int_{-1.2}^{2.3} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \Phi(2.3) - \Phi(-1.2) = 0.8742$$

```
octave:> normcdf(2.3)
ans = 0.9893
octave:> normcdf(-1.2)
ans = 0.1151
octave:> normcdf(2.3)-normcdf(-1.2)
ans = 0.8742
```

Example: Evaluate the integral

$$\frac{1}{\sqrt{2\pi}}\int_{0}^{1.4} e^{-x^2/2}\,dx$$

Example: Evaluate the integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1.1} e^{-x^2/2}dx$$

Example: Evaluate the integral

$$\frac{1}{\sqrt{2\pi}}\int_{-0.5}^{0.5} e^{-x^2/2}\,dx$$

Example: Let X be a random variable with the standard normal distribution. Find:

(a) $P(0<X<1.4)$

(b) $P(X<-1.1)$

(c) $P(|X|<0.5)$

(d) $P(X>3)$

(e) $P(X>5)$

If a random varibale *X* is normally distributed, we compute the probability that *X* lies between *a* and *b* as follows.

1. First we change *a* and *b* into standard units:

$$a^* = \frac{a-\mu}{\sigma} \qquad b^* = \frac{b-\mu}{\sigma}$$

2. Then we compute the probability from

$$P(a < X < b) = P(a^* < X^* < b^*)$$

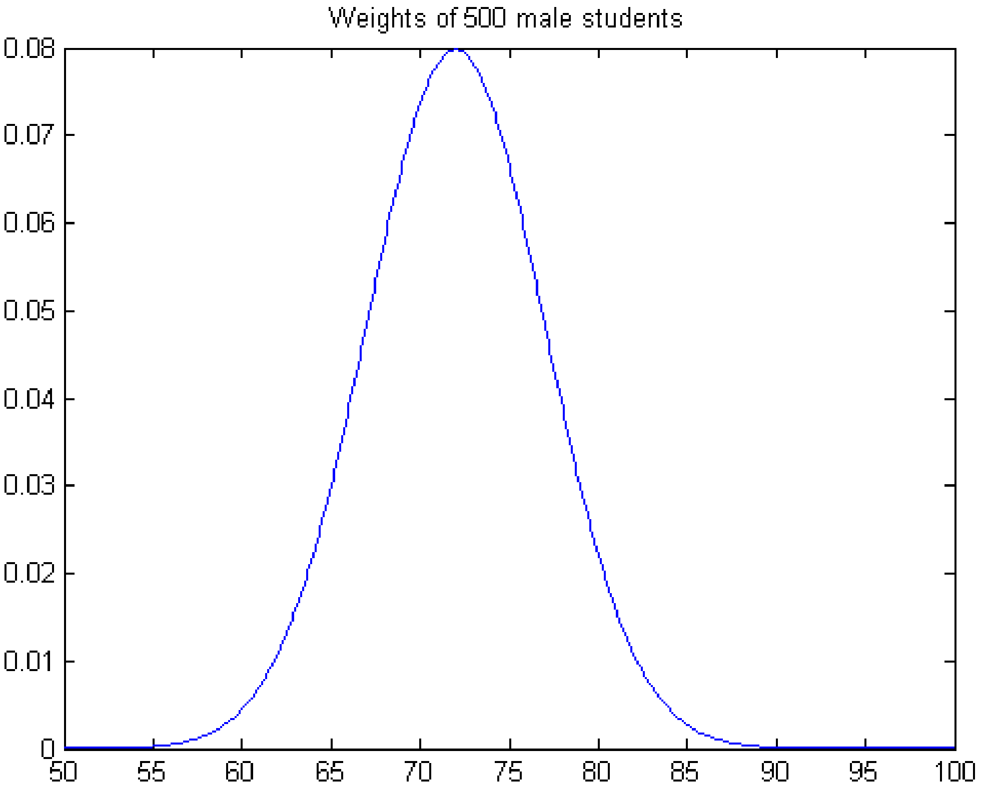
= area under the standard normal curve between a and b

Example: Suppose the temperature during June is normally distributed with mean 20 ^{o}C and standard deviation 3.33 ^{o}C. Find the probability *p* that the temperature is between 21.11 ^{o}C and 26.66 ^{o}C (Answer: 0.3479)

Example: Mean weight of 500 male students at a certain university is 72 kg and the standard deviation is 5 kg. Assuming that the weights are normally distributed, find how many students weigh:

(a) between 66 and 75 kg (Answer: 305)

(b) more than 80 kg (Answer: 27)



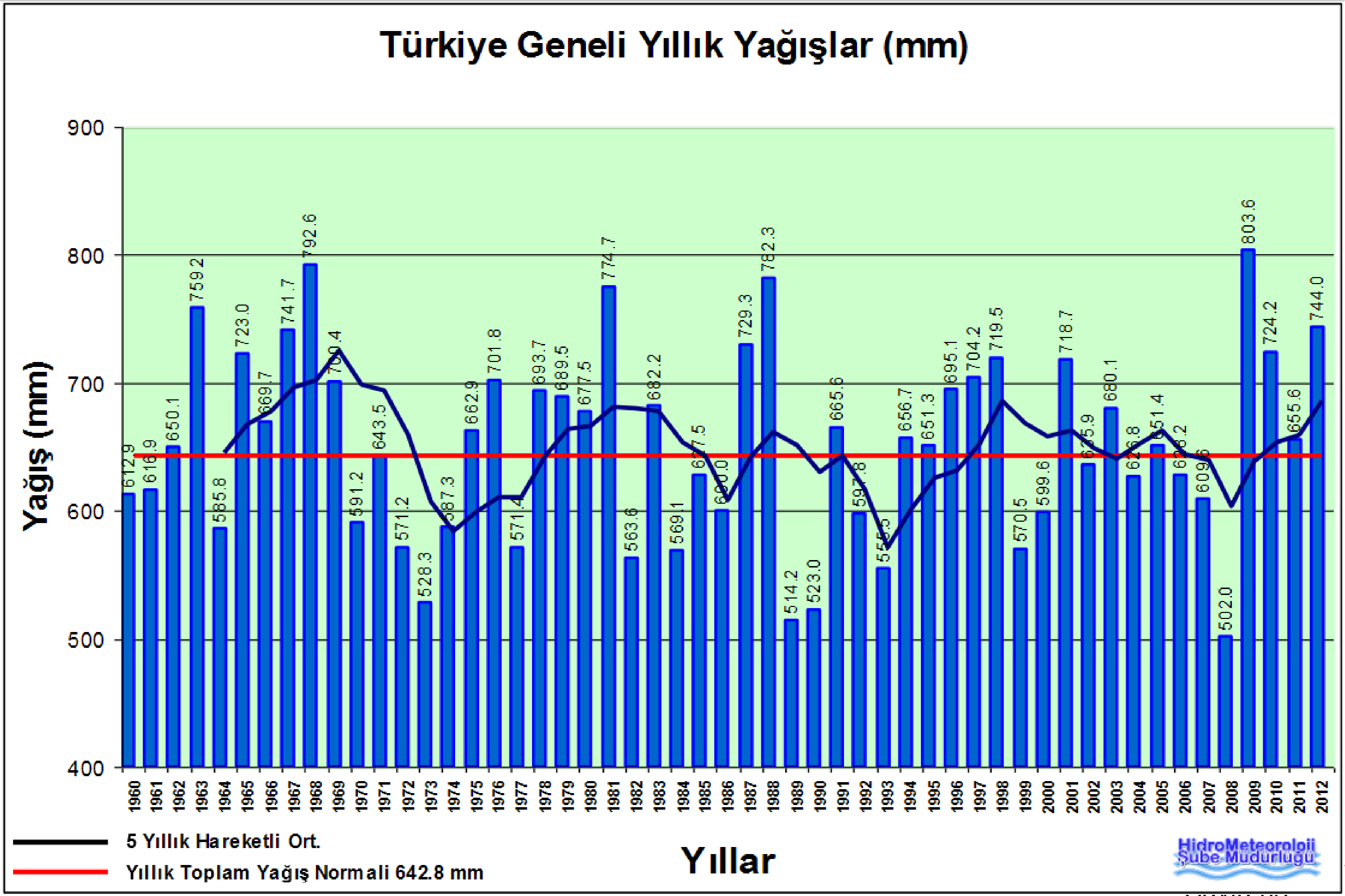
Example Normal Distributions

- Here we will examine some interesting **real data** whose values are distributed normally.

- For each example, histogram of the data is fitted to a Gaussian Function indicated by a blue line.

- All data files can be found at: http://www1.gantep.edu.tr/~bingul/ep122/data

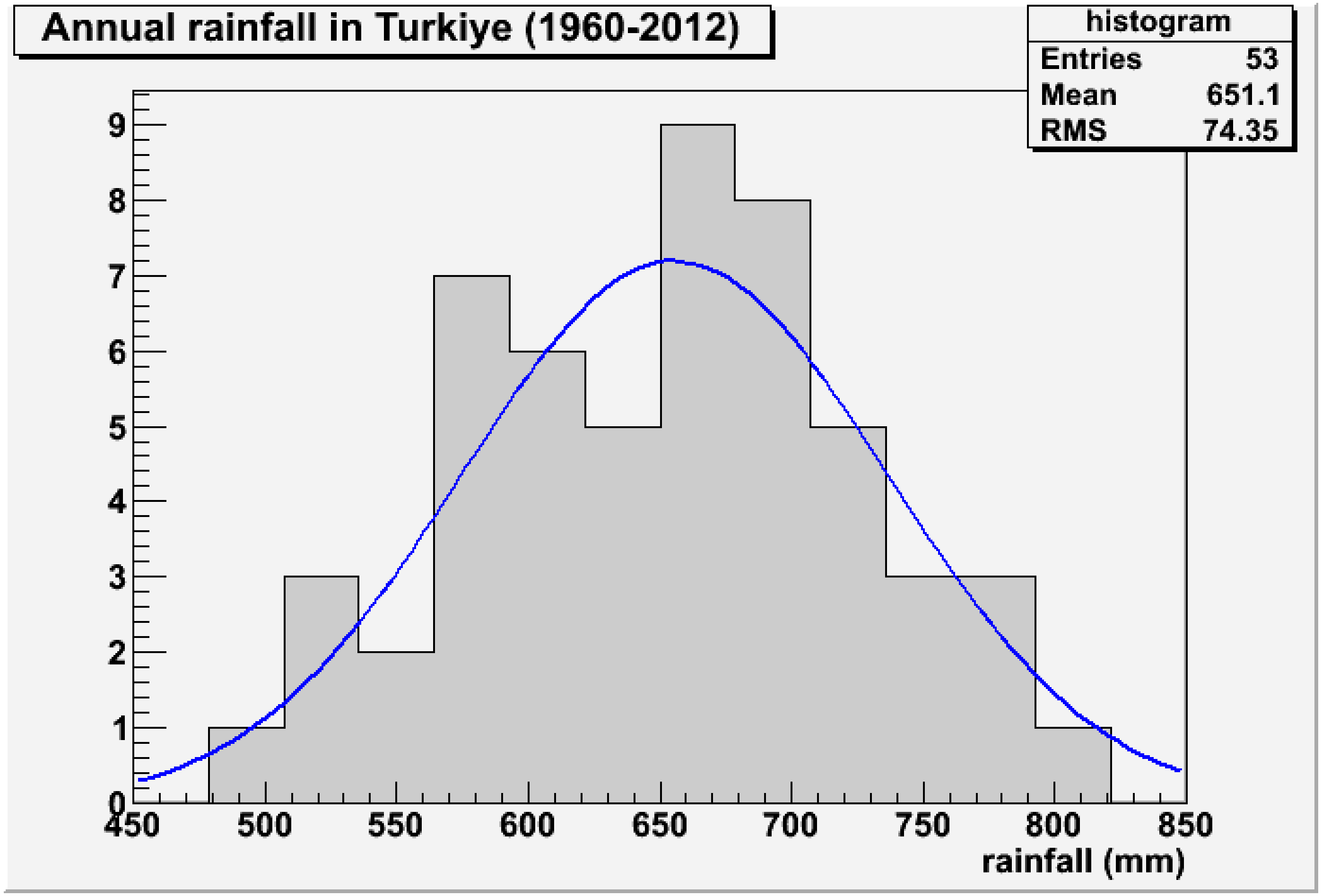
Annual Rainfall (1960-2012)



Annual Rainfall (1960-2012)

Mean : <x> = 651.10 mm

Std. Dev. : σ = 74.35 mm

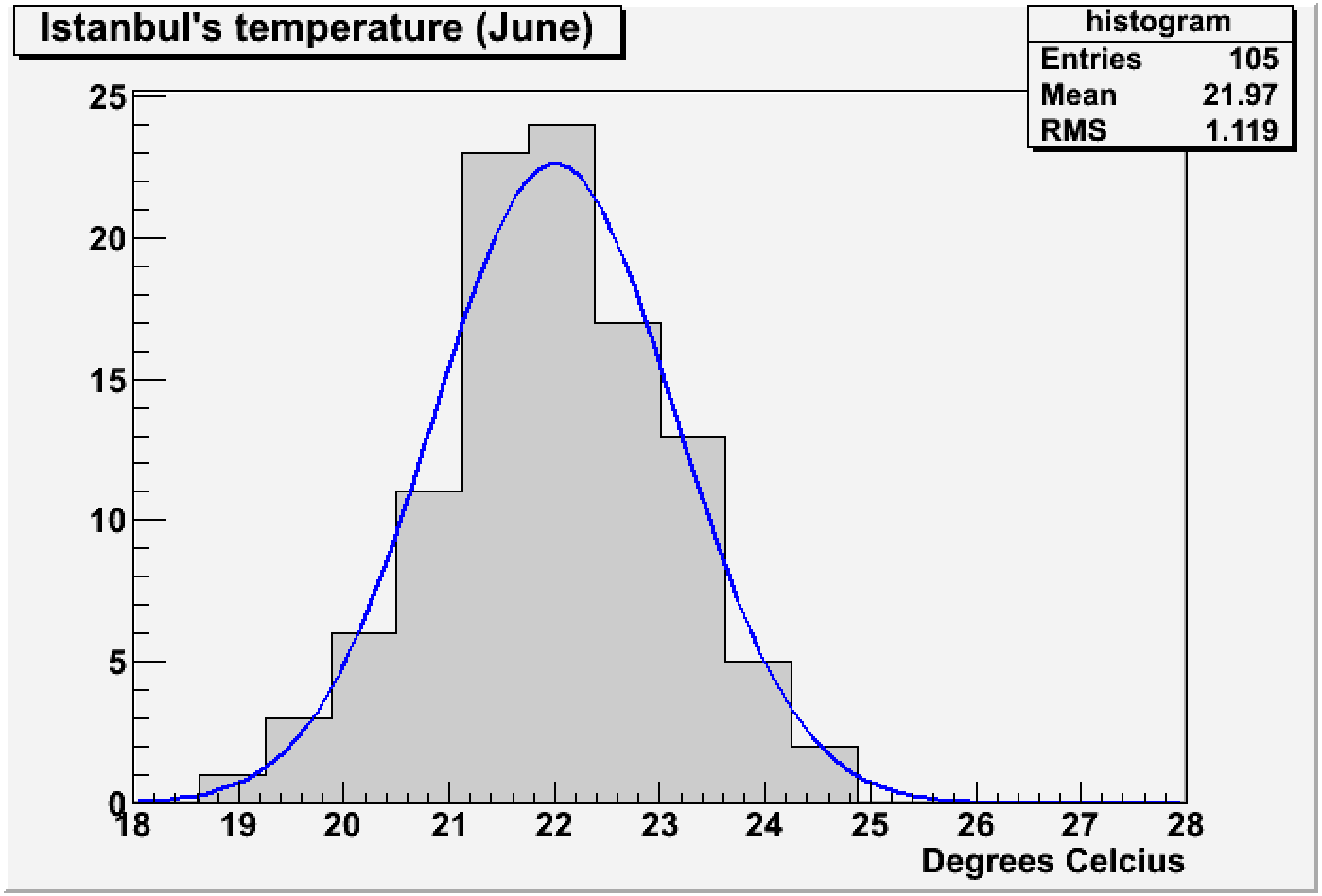


Data: http://www.mgm.gov.tr

Air temperature in Istanbul for the last 105 years.

Mean temperature: <x> = 21.97 ^{o}C

Std. Dev. : σ = 1.12 ^{o}C

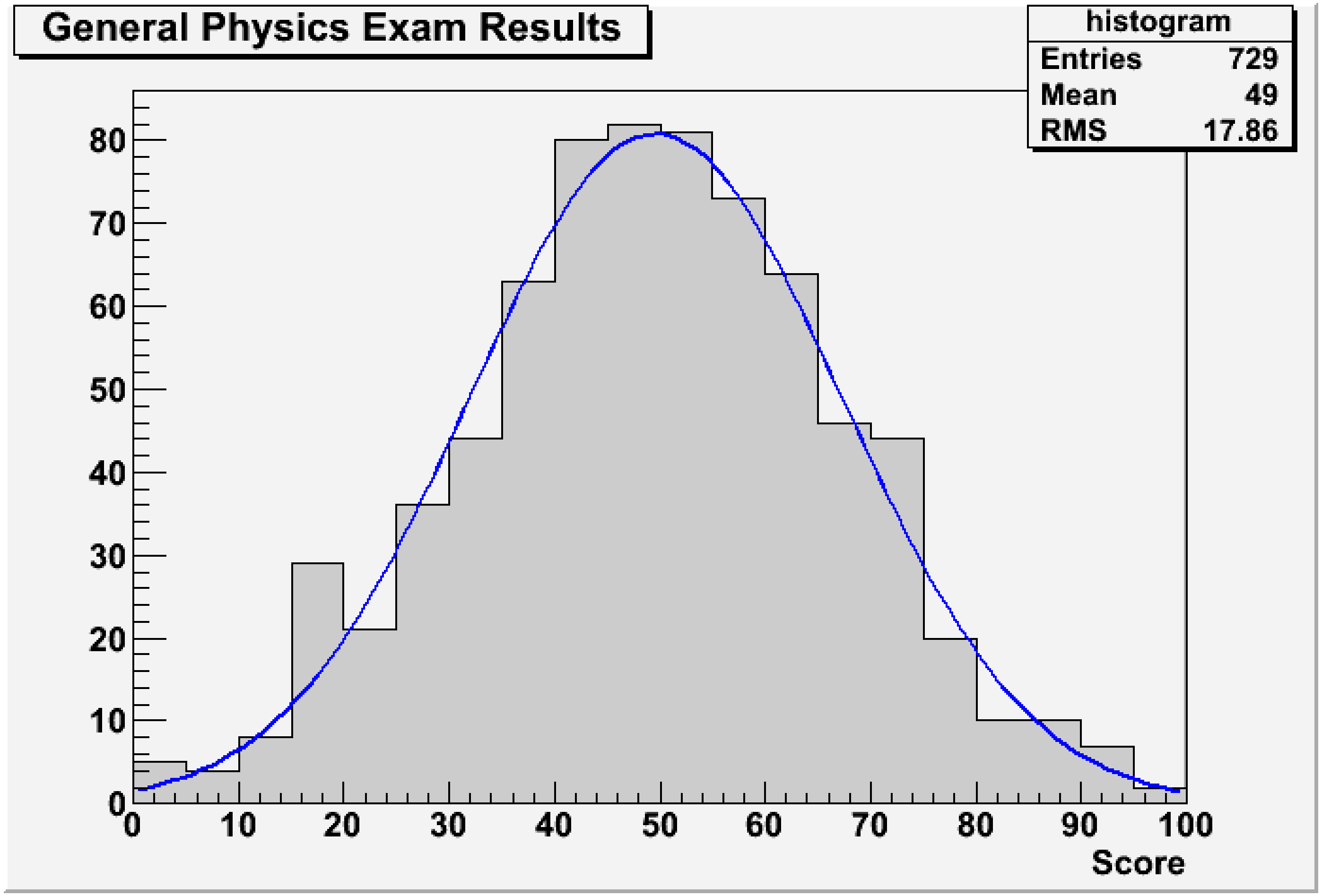


Data: http://data.giss.nasa.gov/tmp/gistemp/STATIONS/tmp_649170620000_14_0/station.txt

“EP106 General Physics II” Course exam results (2010)

Mean score: <x> = 49.0

Std. Dev. : σ = 17.9

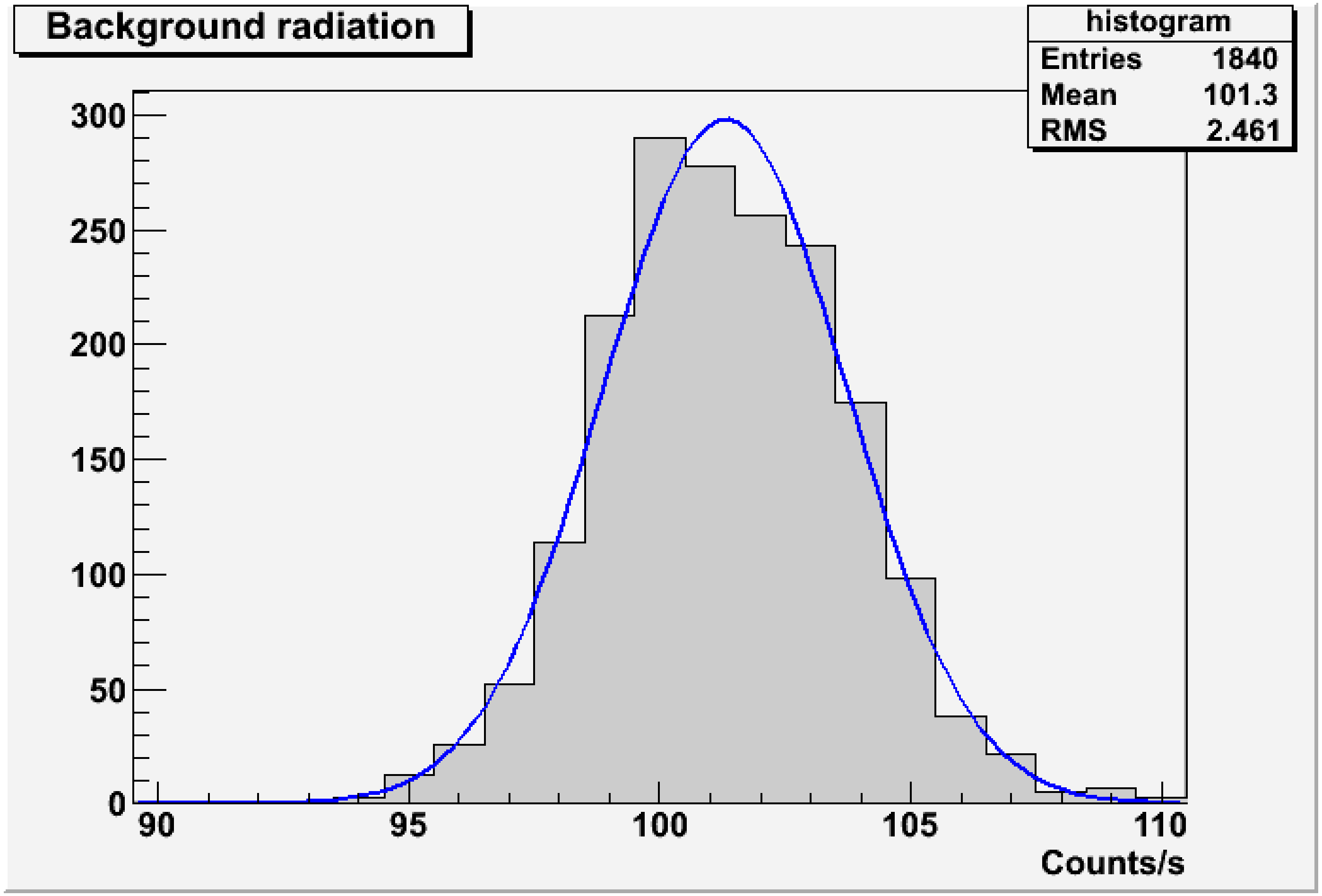


Data: http://www1.gantep.edu.tr/~physics/ep106/exam-statistics.php

Background Radiation in Gaziantep (2013)

Mean : $\langle x \rangle = 101.3$ counts / sec

Std. Dev. : $\sigma = 2.5$ counts / sec



Data is obtained by: Research Assistant Sadık Zuhur (University of Gazaintep)

Questions

1. Let
   U = {a,b,c,d,e,f,g}, A = {a,b,c,d,e}, B = {a,c,e,g}, C = {b,e,f,g}
   Find AnB, AUB, C\B, $(A \backslash B^c)^c$ and C^c n A.

2. Which of the following is a random experiment?
   (a) Tossing a coin.
   (b) Rolling a single 6-sided die.
   (c) Choosing a marble from a jar.
   (d) All of the above.
3. Which of the following is an outcome?
   (a) Rolling a pair of dice.
   (b) Landing on red.
   (c) Choosing 2 marbles from a jar.
   (d) None of the above.
4. Which of the following experiments does NOT have equally likely outcomes?
   (a) Choose a number at random from 1 to 7.
   (b) Toss a coin.
   (c) Choose a letter at random from the word SCHOOL.
   (d) None of the above.

5. What is the probability of choosing a vowel from the English alphabet?
6. A number from 1 to 11 is chosen at random. What is the probability of choosing an odd number?
7. What is the probability of choosing a king from a standard deck of playing cards?
8. What is the probability of choosing the letter i from the word probability?
9. What is the probability of choosing a jack or a queen from a standard deck of 52 playing cards?
10. What is the sample space for choosing a letter from the word mathematics?
11. What is the sample space for choosing a prime number less than 15 at random?
12. What is the probability that a single throw of a die will result in either 2 or 5?

13. A fair coin is tossed 5 times. Find the probability of getting (a) exactly four heads (b) at least two heads (c) no heads.

14. A fair dice is tossed 7 times. Find the probability of getting: (a) exactly four ONEs (b) at least four ONEs (c) no ONEs

15. A woman has 8 children, the probability of each child being female is 50%. What is the probability of being
(a) 4 children female (b) all children female.

16. A communication system contains 6 stations. Independent probability of each station being functional is 90%. If the system requires at least 4 stations to be functional what is the probability that the communication system is functional. (Answer: 0.9842)

17. Suppose 2% of the people on the average are left-handed. Find the probability of 3 or more left-handed among 100 people.
(Answer: 0.325)

18. Given the random variable and corresponding probability mass function (pmf)

$$X = \{1, 2, 3, 4, 5\}$$
$$f(x) = \{0.1, 0.3, 0.4, 0.1, 0.1\}$$

Calculate

(a) $\sum f(x_i)$

(b) $P(1<x<5)$

(c) $E[X]$

(d) $E[X^2]$

(e) RMS

(f) Variance

(g) standard deviation

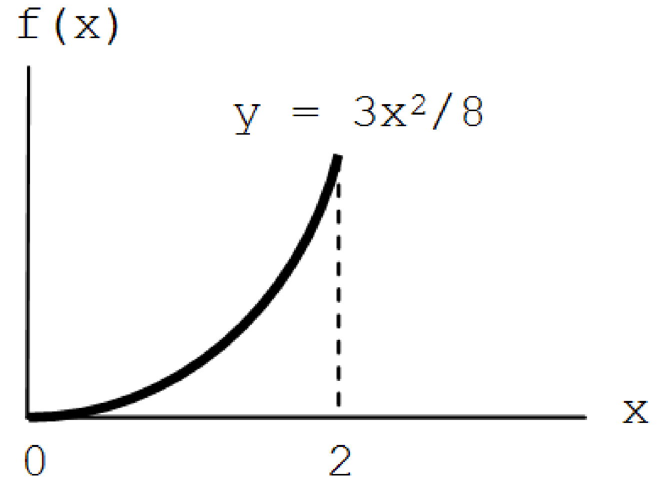
19. Given the probability density function $f(x) = 3x^2/8$.

Calculate

(a) Mean

(b) RMS

(c) $P(x>0.5)$



20. Given the pdf $f(x) = N\sin(x)$ for $0<x<\pi$.

(a) Compute the normalization constant N.

(b) Compute the expectation value and variance.

21. Given the pdf $f(x) = k\exp(-0.1x)$ for $0 < x <$ inf.

(a) Compute the normalization constant k.

(b) Find $P(x>2)$.

22. Evaluate the following integrals:

$$\frac{1}{\sqrt{2\pi}}\int_{1}^{2} e^{-z^2/2}\,dz$$

$$\frac{1}{2\sqrt{2\pi}}\int_{1}^{2} e^{-(z-1)^2/8}\,dz$$

23. Suppose the diameters, D, of screws manufactured by a company are normally distributed with mean 0.25 cm and standard deviation 0.02 cm. A screw is considered defective if its diameter $D < 0.22$ cm.

A sample of 250 screws are selected randomly.

Estimate the number of defective screws in this sample.

References

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2. “Probability and Statistics” - M. Spiegel et. al., Shaum
3. “Probability” - S. Lipshutz, Shaum
4. “Radiation Detection and Measurement”, G.F.Knoll, Wiley
5. http://www1.gantep.edu.tr/~andrew/eee283