Topic 10

Boundary Value Problems

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Introduction

- In mathematics, a **Boundary Value Problem** is a differential equation together with a set of additional restraints, called the boundary conditions.

- A *solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions*.

- See Also:
  
Temperature Distribution in a Rod

Temperature distribution in a rod is given by the following differential equation (DE) if the rod is not insulated:

\[
\frac{d^2T}{dx^2} + k(T_a - T) = 0
\]

where

- \(k\) is heat transfer coefficient (m\(^{-2}\)) and
- \(T_a\) is the temperature of surrounding air in °C.

To obtain a solution there must be appropriate boundary conditions[1]:

\[ T(0) = T_0 \quad \text{and} \quad T(L) = T_L \]

For \(L = 10\) m, \(T_a = 20\) oC, \(T(0) = 40\) oC, \(T(L) = 200\) oC and \(k = 0.01\) m\(^{-2}\)

the analytic solution is:

\[ T(x) = 73.4523 \exp(0.1x) - 53.4523 \exp(-0.1x) + 20.0 \]
We can solve DE using a finite difference approach:

\[ \frac{d^2 T}{dx^2} \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} \]

\[ \frac{d^2 T}{dx^2} + k(T_a - T) = 0 \quad \Rightarrow \quad \frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} + k(T_a - T_i) = 0 \]

or

\[ T_{i+1} - (2 + k(\Delta x)^2)T_i + T_{i-1} = -k(\Delta x)^2 T_a \]

or

\[ -T_{i-1} + (2 + k(\Delta x)^2)T_i - T_{i+1} = k(\Delta x)^2 T_a \]

Using the given values:

\[ -T_{i-1} + 2.04T_i - T_{i+1} = 0.8 \]
\[ T_1 = T(0) = 40 \]

\[ -T_{i-1} + 2.04T_i - T_{i+1} = 0.8 \]

for \( i = 2 \): \(-40 + 2.04T_2 - T_3 = 0.8\)

for \( i = 3 \): \(-T_2 + 2.04T_3 - T_4 = 0.8\)

for \( i = 4 \): \(-T_3 + 2.04T_4 - T_5 = 0.8\)

for \( i = 5 \): \(-T_4 + 2.04T_5 - 200 = 0.8\)

\[ \Delta x = \frac{L}{5} = 2 \text{ m} \]

\[ T_6 = T(L) = 200 \]

\[
\begin{pmatrix}
2.04 & -1 & 0 & 0 & \left( T_2 \right)
\end{pmatrix}
= 
\begin{pmatrix}
40.8 \\
-1 & 2.04 & -1 & 0 & \left( T_3 \right)
\end{pmatrix}
= 
\begin{pmatrix}
0.8 \\
0 & -1 & 2.04 & -1 & \left( T_4 \right)
\end{pmatrix}
= 
\begin{pmatrix}
0.8 \\
0 & 0 & -1 & 2.04 & \left( T_5 \right)
\end{pmatrix}
= 
\begin{pmatrix}
200.8
\end{pmatrix}
\]
```matlab
>> A = [2.04 -1 0 0; -1 2.04 -1 0; 0 -1 2.04 -1; 0 0 -1 2.04];
>> b = [40.8 0.8 0.8 200.8];
>> T = A \ b';
T =
65.9698
93.7785
124.5382
159.4795
```
% trod.m
A=[ 2.04 -1.00 0.00 0.00
    -1.00 2.04 -1.00 0.00
    0.00 -1.00 2.04 -1.00
    0.00 0.00 -1.00 2.04];

b = [40.8 0.8 0.8 200.8];

x  = 0:2:10;
Tt = 73.4523*exp(0.1*x)-53.4523*exp(-0.1*x) + 20.0 % theory
Tn = A\b'; % numerical solution
Tn = [40 Tn' 200]

plot(x,Tt)
hold on
plot(x,Tn,'-s', ...
     'MarkerFaceColor','r',...
     'MarkerEdgeColor','k',...
     'MarkerSize',10);
xlabel('x (m)')
ylabel('Temperature oC')
For $N = 10$ parts triangular matrix and temperatures become:

\[
\begin{bmatrix}
 2.01 & -1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.00 & 2.01 & -1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.00 & 2.01 & -1.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1.00 & 2.01 & -1.00 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.00 & 2.01 & -1.00 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1.00 & 2.01 & -1.00 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1.00 & 2.01 & -1.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1.00 & 2.01 & -1.00 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.00 & 2.01 & -1.00 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.00 & 2.01 \\
\end{bmatrix}
\]

*** Temperatures (°C) in the rod for $n = 10$ parts ***

$T(0) = 40.0000$
$T(1) = 52.8141$
$T(2) = 65.9563$
$T(3) = 79.5581$
$T(4) = 93.7555$
$T(5) = 108.6904$
$T(6) = 124.5122$
$T(7) = 141.3792$
$T(8) = 159.4600$
$T(9) = 178.9353$
$T(n) = 200.0000$
Spring-Mass System

A spring-mass system is governed by D.E.

\[
m \frac{d^2 x}{dt^2} + kx + b \frac{dx}{dt} + \mu mg = 0
\]

where

- \(m\) is the mass (kg)
- \(k\) is the spring constant (N/m)
- \(b\) is the coefficient of air friction (kg/s)
- \(\mu u\) is the coefficient of surface kinetic friction (unitless)
- \(x\) is the distance from equilibrium position (m)
- \(t\) is the time (s)

For \(m = 1\), \(k = 10\) and \(\mu u = b = 0.1\)

(a) obtain a solution if \(x(0) = 0.2\) and \(x(2) = 0.3\) by using finite difference form

(b) obtain a solution if \(x(0) = 0.2\) and \(v(0) = 0.0\) by using RK4
References:


[3]. Numerical Methods in Engineering with MATLAB,

[4]. Essential MATLAB for Engineers and Scientist, 3rd Ed

[5]. Computational Physics,