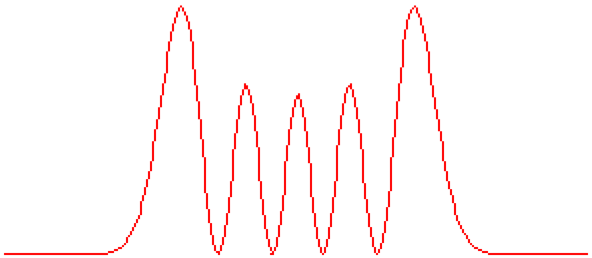
EP547 Computational Methods in QM

Topic 10

Boundary Value Problems



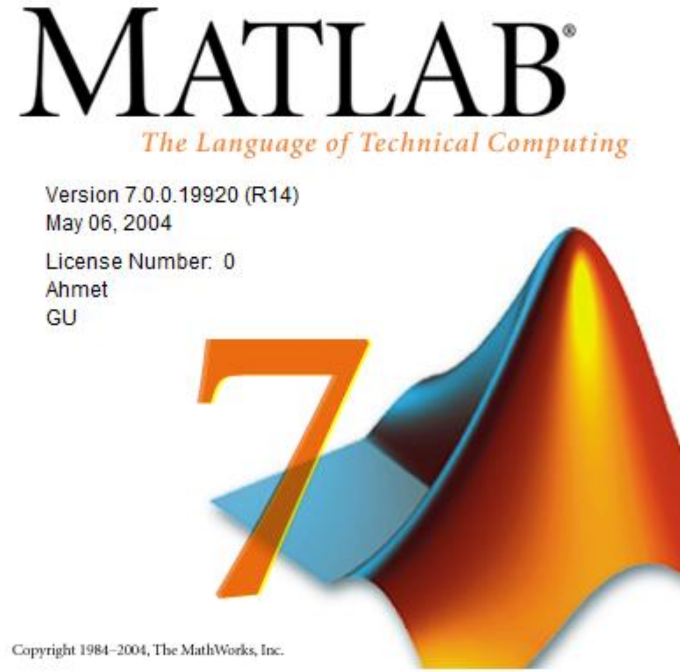
Department of
Engineering Physics
University of Gaziantep

Feb 2013

Content

1. **Introduction**

2. **Boundary Value Problems**



Introduction

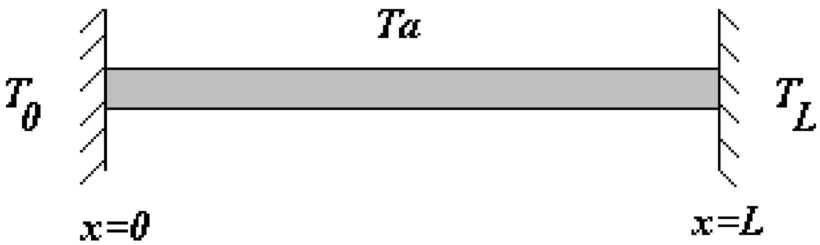
- In mathematics, a **Boundary Value Problem** is a differential equation together with a set of additional restraints, called the boundary conditions.

- *A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions*.

- See Also:
  `http://en.wikipedia.org/wiki/Boundary_value_problem`

Temperature Distribution in a Rod

Temperature distribution in a rod is given by the following differential equation (DE) if the rod is not insulated:



$$\frac{d^2T}{dx^2} + k(T_a - T) = 0$$

where

k is heat transfer coefficient (m^{-2}) and

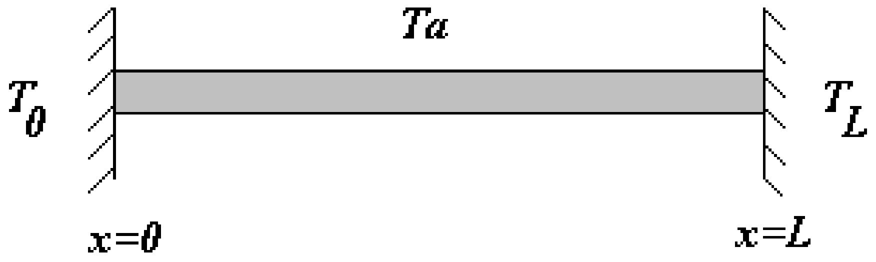
Ta is the temperature of surrounding air in ^oC.

To obtain a solution there must be appropriate boundary conditions[1]:

$$T(0) = T_0 \quad \text{and} \quad T(L) = T_L$$

For L = 10 m, Ta = 20 oC, $T(0)$=40 oC ,$T(L)$ = 200 oC and k = 0.01 m^{-2} the analytic solution is:

$$\mathbf{T(x) = 73.4523 \exp(0.1x) - 53.4523 \exp(-0.1x) + 20.0}$$



We can solve DE using a finite diference approach:

$$\frac{d^2T}{dx^2} \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2}$$

$$\frac{d^2T}{dx^2} + k(T_a - T) = 0 \quad \longrightarrow \quad \frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} + k(T_a - T_i) = 0$$

or

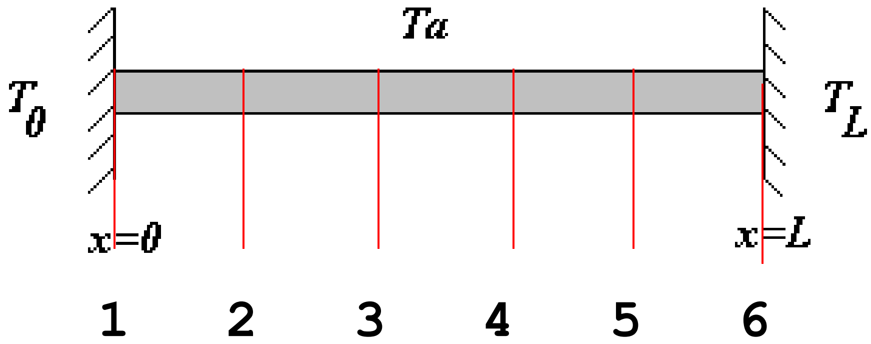
$$T_{i+1} - (2 + k(\Delta x)^2)T_i + T_{i-1} = -k(\Delta x)^2 T_a$$

or

$$-T_{i-1} + (2 + k(\Delta x)^2)T_i - T_{i+1} = k(\Delta x)^2 T_a$$

Using the given values: $-T_{i-1} + 2.04T_i - T_{i+1} = 0.8$

$$T_1 = T(0) = 40$$



$$T_6 = T(L) = 200$$

$$\Delta x = \frac{L}{5} = 2\ \text{m}$$

$$-T_{i-1} + 2.04T_i - T_{i+1} = 0.8$$

$$\text{for } i = 2:\ -40 + 2.04T_2 - T_3 = 0.8$$

$$\text{for } i = 3:\ -T_2 + 2.04T_3 - T_4 = 0.8$$

$$\text{for } i = 4:\ -T_3 + 2.04T_4 - T_5 = 0.8$$

$$\text{for } i = 5:\ -T_4 + 2.04T_5 - 200 = 0.8$$

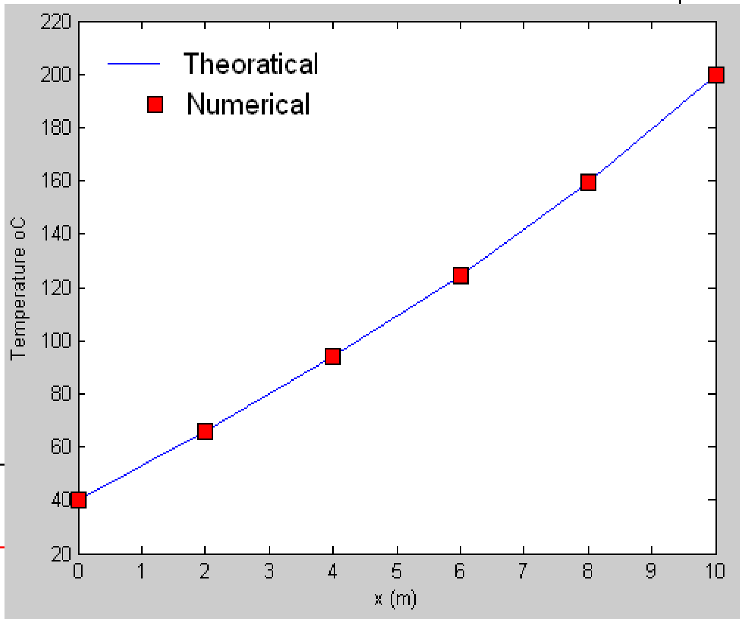
Tridiagonal Matrix

$$\begin{pmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{pmatrix}$$

$$
\begin{pmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{pmatrix}
$$

```
>> A=[ 2.04   -1.00    0.00   0.00
      -1.00    2.04   -1.00   0.00
       0.00   -1.00    2.04  -1.00
       0.00    0.00   -1.00   2.04];
>> b = [40.8 0.8 0.8 200.8];
>> T = A\b'
T =
   65.9698
   93.7785
  124.5382
  159.4795
```

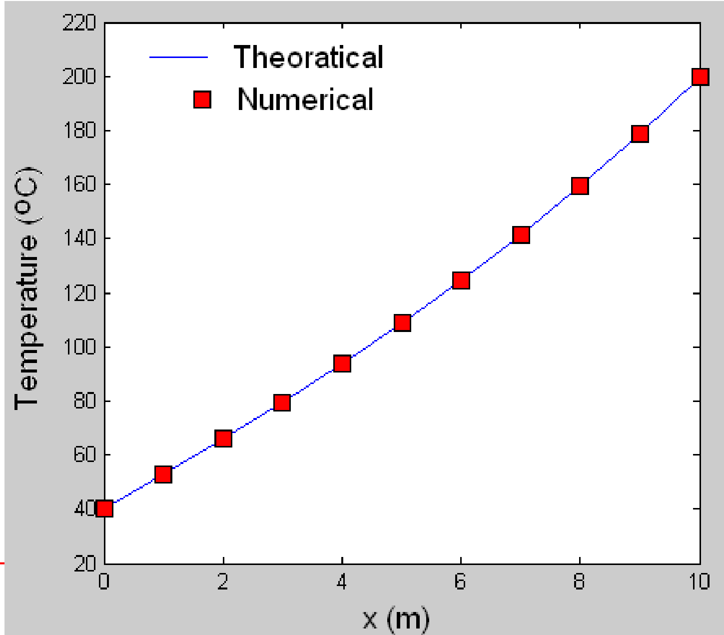
```
% trod.m
A=[ 2.04   -1.00    0.00   0.00
   -1.00    2.04   -1.00   0.00
    0.00   -1.00    2.04  -1.00
    0.00    0.00   -1.00   2.04];

b = [40.8 0.8 0.8 200.8];

x  = 0:2:10;
Tt = 73.4523*exp(0.1*x)-53.4523*exp(-0.1*x) + 20.0 % theory
Tn = A\b'; % numerical solution
Tn = [40 Tn' 200]

plot(x,Tt)
hold on
plot(x,Tn,'-s', ...
          'MarkerFaceColor','r',...
          'MarkerEdgeColor','k',...
          'MarkerSize',10);
xlabel('x (m)')
ylabel('Temperature oC')
```



For N = 10 parts triangular matrix and temperatures become:

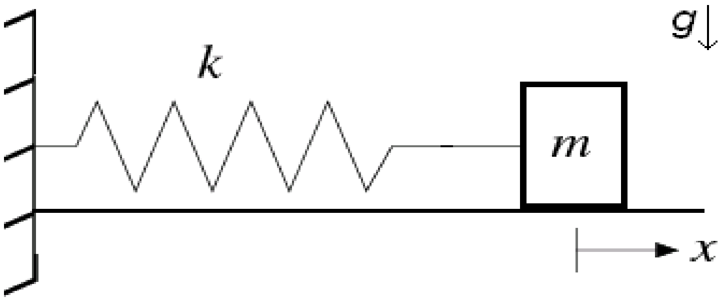
2.01	-1.00	0	0	0	0	0	0	0
-1.00	2.01	-1.00	0	0	0	0	0	0
0	-1.00	2.01	-1.00	0	0	0	0	0
0	0	-1.00	2.01	-1.00	0	0	0	0
0	0	0	-1.00	2.01	-1.00	0	0	0
0	0	0	0	-1.00	2.01	-1.00	0	0
0	0	0	0	0	-1.00	2.01	-1.00	0
0	0	0	0	0	0	-1.00	2.01	-1.00
0	0	0	0	0	0	0	-1.00	2.01

```
*** Temperatures (oC) in the rod for n = 10 parts ***
T( 0) =        40.0000
T( 1) =        52.8141
T( 2) =        65.9563
T( 3) =        79.5581
T( 4) =        93.7555
T( 5) =       108.6904
T( 6) =       124.5122
T( 7) =       141.3792
T( 8) =       159.4600
T( 9) =       178.9353
T( n) =       200.0000
```



Spring-Mass System

A spring-mass system is governed by D.E.



$$m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} + \mu\, mg = 0$$

where

m	is the mass (kg)
k	is the spring constant (N/m)
b	is the coefficient of air friction (kg/s)
mu	is the coefficient of surface kinetic friction (unitless)
x	is the distance from equilibrium position (m)
t	is the time (s)

For *m* = 1, *k* =10 and *mu* = b = 0.1

(a) obtain a solution if *x*(0) = 0.2 and *x*(2) = 0.3 by using finite difference form

(b) obtain a solution if *x*(0) = 0.2 and *v*(0) = 0.0 by using RK4

References:

[1]. Numerical Methods for Engineers, 6th Ed.
S.C. Chapra, Mc Graw Hill (2010)

[2]. http://www.mathworks.com/products/matlab

[3]. Numerical Methods in Engineering with MATLAB,
J. Kiusalaas, Cambridge University Press (2005)

[4]. Essential MATLAB for Engineers and Scientist, 3rd Ed
Hahn B., Valentine D.T. (2007)

[5]. Computational Physics,
Giordano J.N. Prentice Hall (1997)