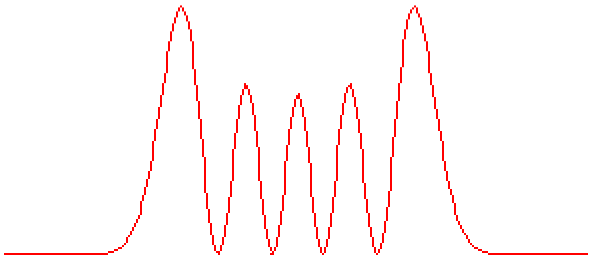
EP547 Computational Methods in QM

Topic 7

Solutions of Linear Equations



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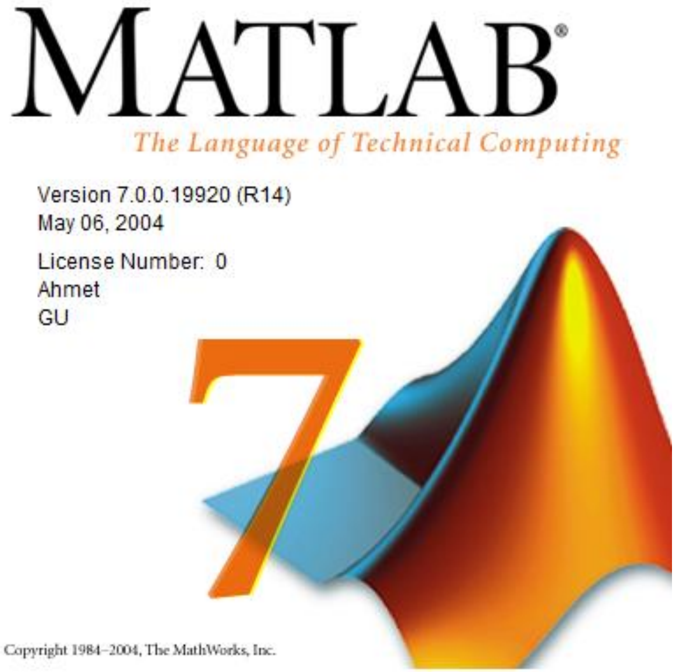


Linear Algebraic Equations

- Mathematical formulations of engineering problems often lead to sets of simultaneous linear equations of the form:

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{n,1}x_1 + a_{n,2}x_2 + \cdots a_{n,n}x_n &= b_n
\end{aligned}
$$

This system can be written in matrix form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

or simply:

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

Here

A is called the "coefficient matrix"

x is the "unknown vector"

b is the "constant vector" or "right hand side vector"

The solution of the system

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

is

$$\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{b}$$

where $\mathbf{A}^{-1} = \mathrm{adj}(\mathbf{A}) / |\mathbf{A}|$

Here

adj(**A**) is adjoint of the matrix A

|**A**| is the determinant of A.

Note that to get the solution the condition:

$$|\mathbf{A}| \neq 0$$

must be satisfied.

Example 1: Consider the equation:

$$x + y = 4$$
$$x - 3y = 0$$

which can be written in matrix form: $\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

The solution is: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

Here: $\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}^{-1} = \frac{1}{-4}\begin{pmatrix} -3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & -1/4 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\longrightarrow$ $x = 3 , y = 1$

- MATLAB solution:

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

```
>> A = [1 1; 1 -3];   % coefficient matix
>> b = [4 ; 0];       % row vector
>> x = A\b            % solution vector
x =   3
      1
```

```
>> A = [1 1; 1 -3];   % coefficient matix
>> b = [4 0]';        % row vector
>> x = A\b            % solution vector
x =   3
      1
```

```
>> A = [1 1; 1 -3];   % coefficient matix
>> b = [4 0]';        % row vector
>> x = inv(A)*b       % solution vector
x =   3
      1
```

Example 2: Consider the equation:

$$\begin{aligned} x &+ y + z = 6 \\ -2x &+ y = 0 \\ 3x &+ 2y + z = 10 \end{aligned}$$

which can be written in matrix form:

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 10 \end{pmatrix}$$

```
>> A = [1 1 1; -2 1 0; 3 2 1];
>> b = [6 0 10]';
>> x = A\b
x =  1
     2
     3
```

Example 3: Consider the equation:

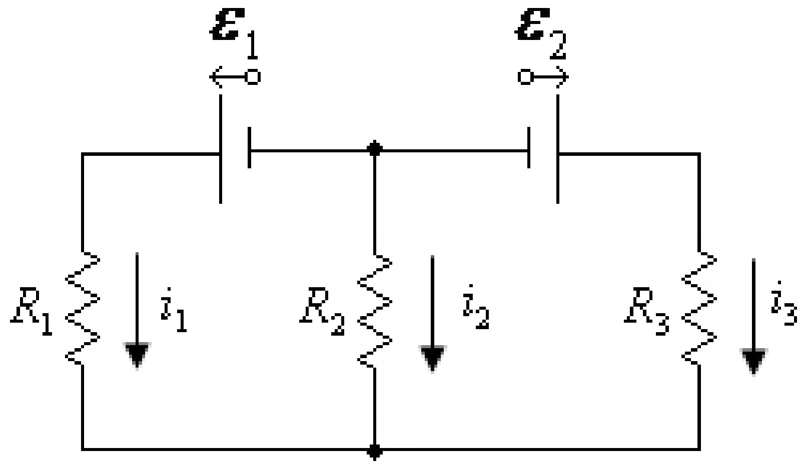
$$
\begin{aligned}
6x_1 - x_3 &= 50 \\
3x_1 - 3x_2 &= 0 \\
-x_2 + 9x_3 &= 160 \\
x_2 + 8x_3 - 11x_4 + 2x_5 &= 0 \\
3x_1 + x_2 - 4x_5 &= 0
\end{aligned}
$$

$$
\begin{pmatrix} 6 & 0 & -1 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 \\ 0 & -1 & 9 & 0 & 0 \\ 0 & 1 & 8 & -11 & 2 \\ 3 & 1 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 160 \\ 0 \\ 0 \end{pmatrix}
$$

```
>> A = [ 6   0  -1   0  0;
         3  -3   0   0  0;
         0  -1   9   0  0;
         0   1   8 -11  2;
         3   1   0   0 -4];

>> b = [50 0 160 0 0]';
>> x = inv(A)*b
x =
   11.5094
   11.5094
   19.0566
   16.9983
   11.5094
```

Example 4: Consider the circuit:



Two batteries with ε_1 = 3 V and ε_2 = 5 V are connected with three resistors R_1=10 Ω, R_2 = 20 Ω and R_3 = 30 Ω.

Using Kirchhoff's laws, find the currents i_1, i_2 and i_3 passing through the resistors.

Eigenvalues

```
>> A = [1 3 -2; 3 5 1; -2 1 4]
A =
     1     3    -2
     3     5     1
    -2     1     4

>> [v d] = eig(A)

v =
   -0.8184   -0.3153   -0.4804
    0.4347    0.2071   -0.8764
   -0.3758    0.9261    0.0324

d =
   -1.5120         0         0
         0    4.9045         0
         0         0    6.6076
```

References

[1]. http://www.mathworks.com/products/matlab

[2]. Numerical Methods in Engineering with MATLAB,
J. Kiusalaas, Cambridge University Press (2005)

[3]. Numerical Methods for Engineers, 6th Ed.
S.C. Chapra, Mc Graw Hill (2010)