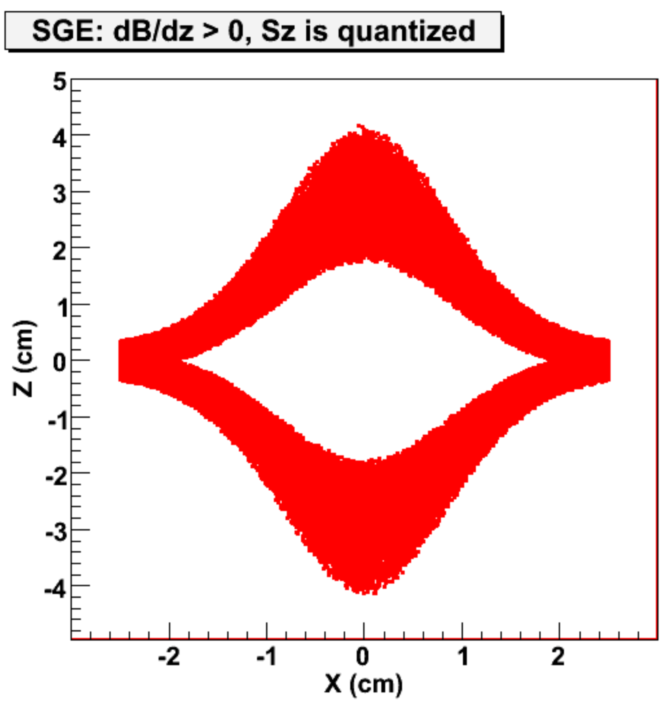
A Monte-Carlo Simulation of the Stern-Gerlach Experiment



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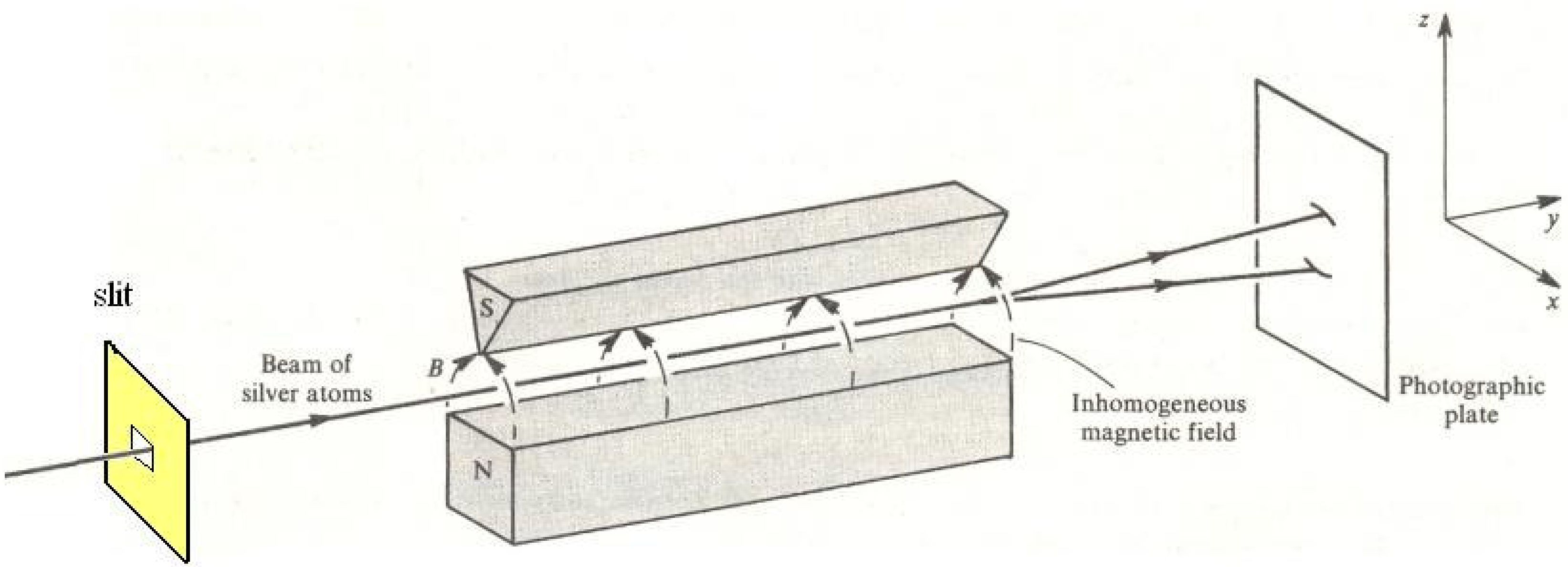
Content

- Stern-Gerlach Experiment (SGE)
- Electron spin
- Monte-Carlo Simulation

You can find the slides of this seminar and computer programs at:

http://www1.gantep.edu.tr/~bingul/seminar/spin

The Stern-Gerlach Experiment



- The Stern-Gerlach Experiment (SGE) is performed in 1921, to see if electron has an intrinsic magnetic moment.
- A beam of hot (neutral) **Silver** ($_{47}$Ag) atoms was used.
- The beam is passed through an *inhomogeneous* magnetic field along z axis. This field would interact with the magnetic dipole moment of the atom, if any, and deflect it.
- Finally, the beam strikes a photographic plate to measure, if any, deflection.

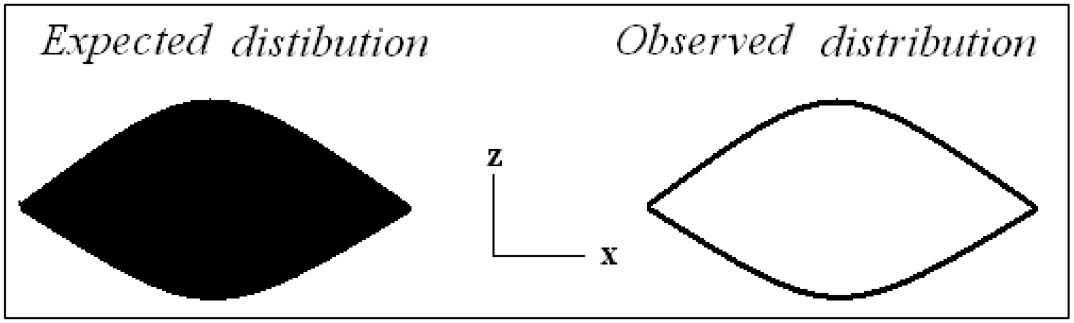
The Stern-Gerlach Experiment

- **Why Neutral Silver atom?**
  - No Lorentz force (**F** = q**v** x **B**) acts on a neutral atom, since the total charge (q) of the atom is zero.
  - Only the magnetic moment of the atom interacts with the external magnetic field.
  - Electronic configuration:
    $1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^6\ 3d^{10}\ 4s^1\ 4p^6\ 4d^{10}\ 5s^1$
    So, a neutral Ag atom has zero total orbital momentum.
  - Therefore, if the electron at 5s orbital has a magnetic moment, one can measure it.

- **Why inhomogenous magnetic Field?**
  - In a **homogeneous** field, each magnetic moment experience only a torque and no deflecting force.
  - An **inhomogeneous** field produces a deflecting force on any magnetic moments that are present in the beam.

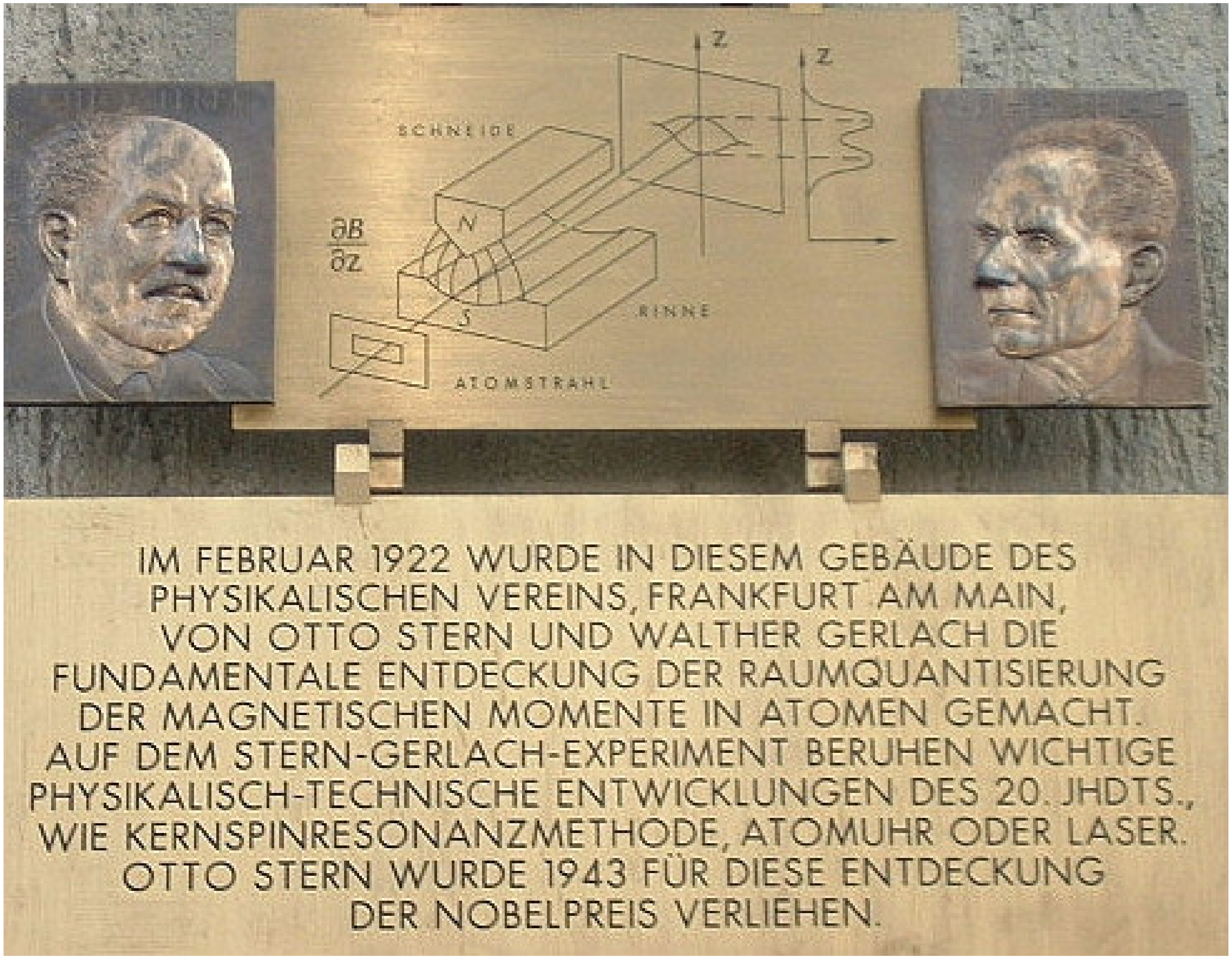
The Stern-Gerlach Experiment

- In the experiment, they saw a deflection on the photographic plate. Since atom has zero total magnetic moment, *the magnetic interaction producing the deflection should come from another type of magnetic field*. That is to say: electron's (at 5s orbital) acted like a bar magnet.
- If the electrons were like ordinary magnets with random orientations, they would show a continues distribution of pats. The photographic plate in the SGE would have shown a continues distribution of impact positions.
- However, in the experiment, it was found that the beam pattern on the photographic plate had split into two distinct parts. Atoms were deflected either up or down by a constant amount, in roughly equal numbers.
- ***Apparently, z component of the electron's spin is quantized.***



The Stern-Gerlach Experiment

A plaque at the Frankfurt institute commemorating the experiment

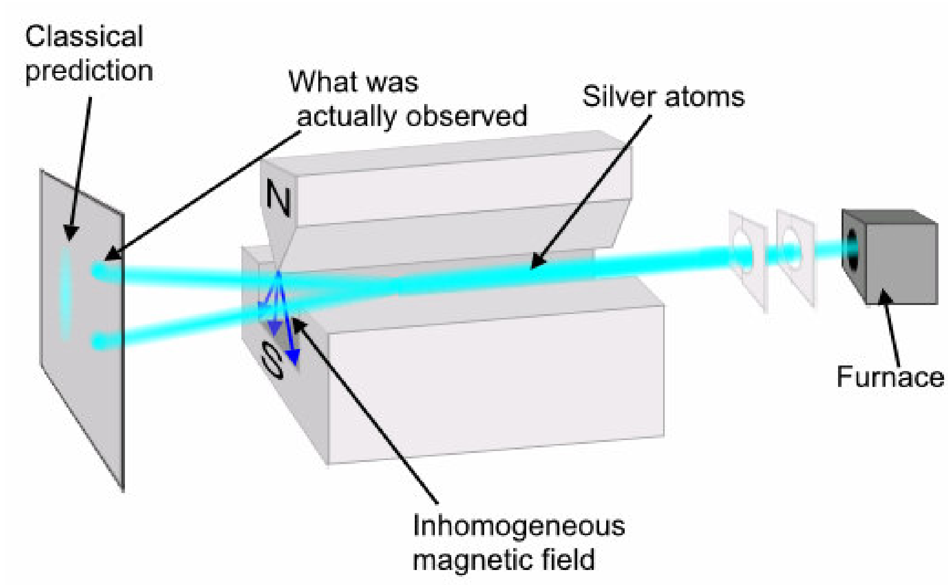


Electron Spin

- 1925: S.A Goutsmit and G.E. Uhlenbeck suggested that an *electron has an intrinsic angular momentum* (i.e. magnetic moment) called its **spin**.

- The extra magnetic moment $\boldsymbol{\mu_s}$ associated with angular momentum **S** accounts for the deflection in SGE.

- Two equally spaced lined observed in SGE shows that electron has two orientations with respect to magnetic field.

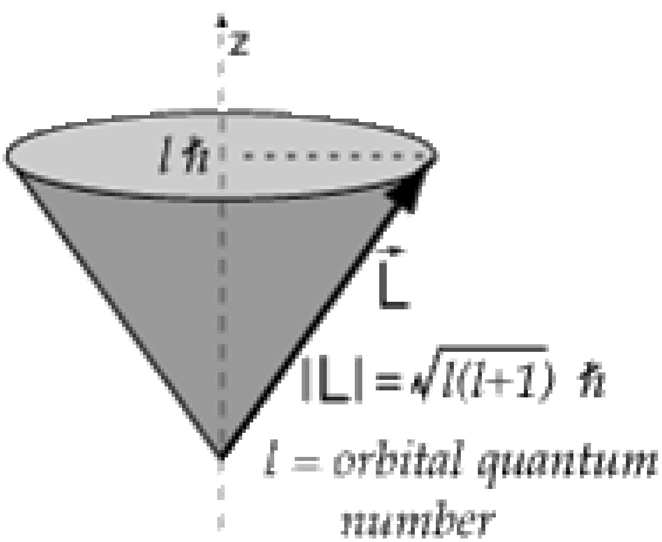


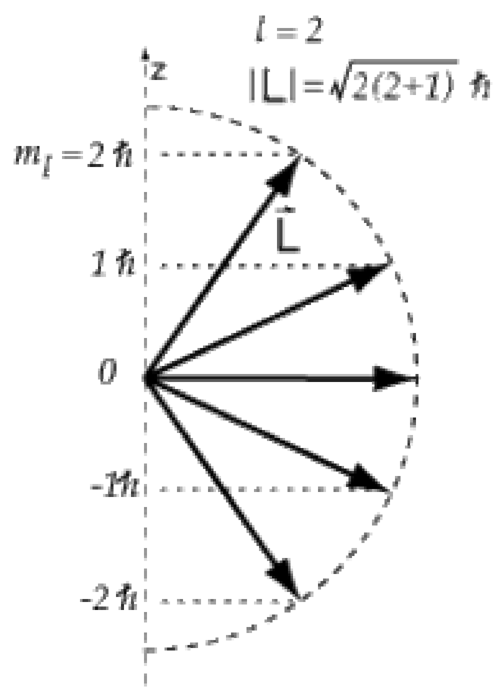
Electron Spin

- Orbital motion of electrons, is specified by the quantum number l.
- Along the magnetic field, l can have $2l+1$ discrete values.

$$L = \sqrt{l(l+1)}\hbar \qquad l = 0,1,2,\cdots,n-1$$

$$L_z = m_l\hbar \qquad m_l = l, l-1,\cdots,-(l-1),-l$$

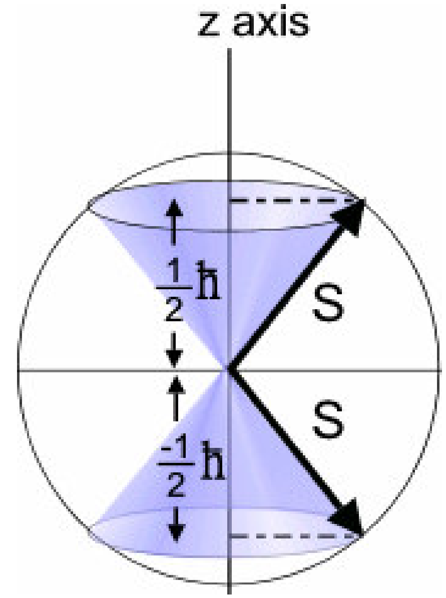




Electron Spin

- Similar to orbital angular momentum **L**, the spin vector **S** is quantized both in magnitude and direction, and can be specified by spin quantum number s.

- We have two orientations: $2 = 2s+1 \rightarrow s = 1/2$



$$S = \sqrt{s(s+1)}\hbar = \sqrt{1/2(1/2\ +1)}\hbar = \frac{\sqrt{3}}{2}\hbar$$

The component S_z along z axis:

$$S_z = m_s\hbar \qquad \begin{array}{ll} m_s = +1/2 & \text{(spin up)} \\ m_s = -1/2 & \text{(spin down)} \end{array}$$

Electron Spin

It is found that intrinsic magnetic moment ($\boldsymbol{\mu}_s$) and angular momentum (**S**) vectors are proportional to each other:

$$\boldsymbol{\mu}_s = -g_s \frac{e}{2m} \mathbf{S}$$

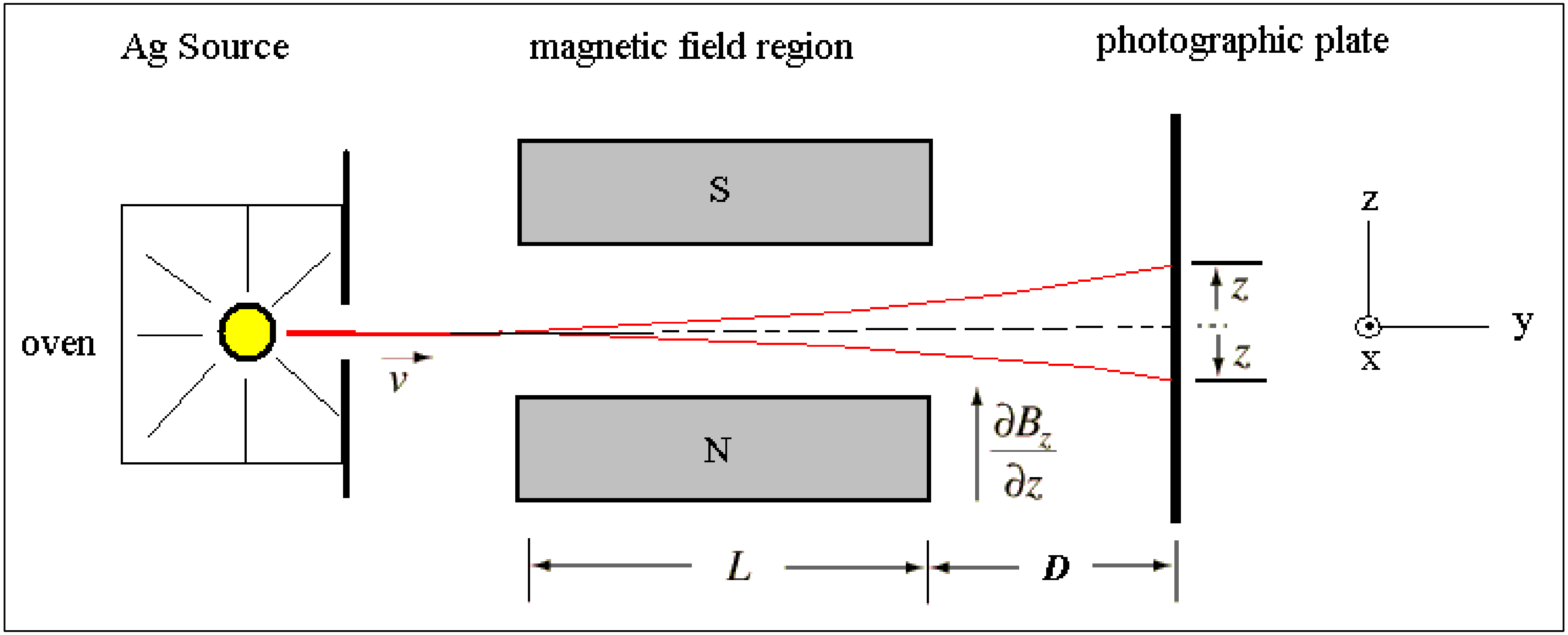
where g_s is called *gyromagnetic* ratio.

For the electron, $g_s = 2.0023$.

The properties of electron spin were first explained by Dirac (1928), by combining quantum mechanics with theory of relativity.

Monte-Carlo Simulation

Experimental Set-up:



Monte-Carlo Simulation

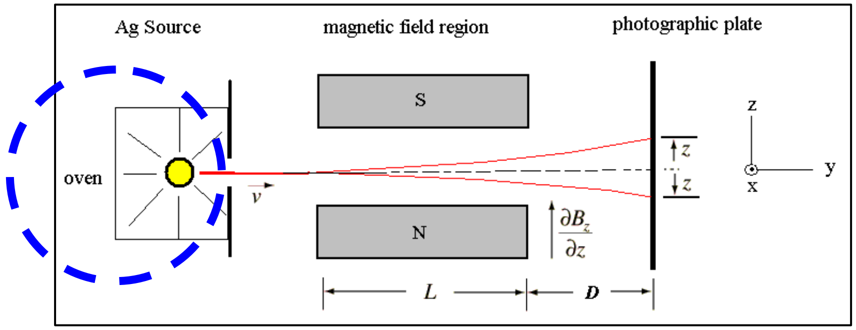
Ag atoms and their velocities:

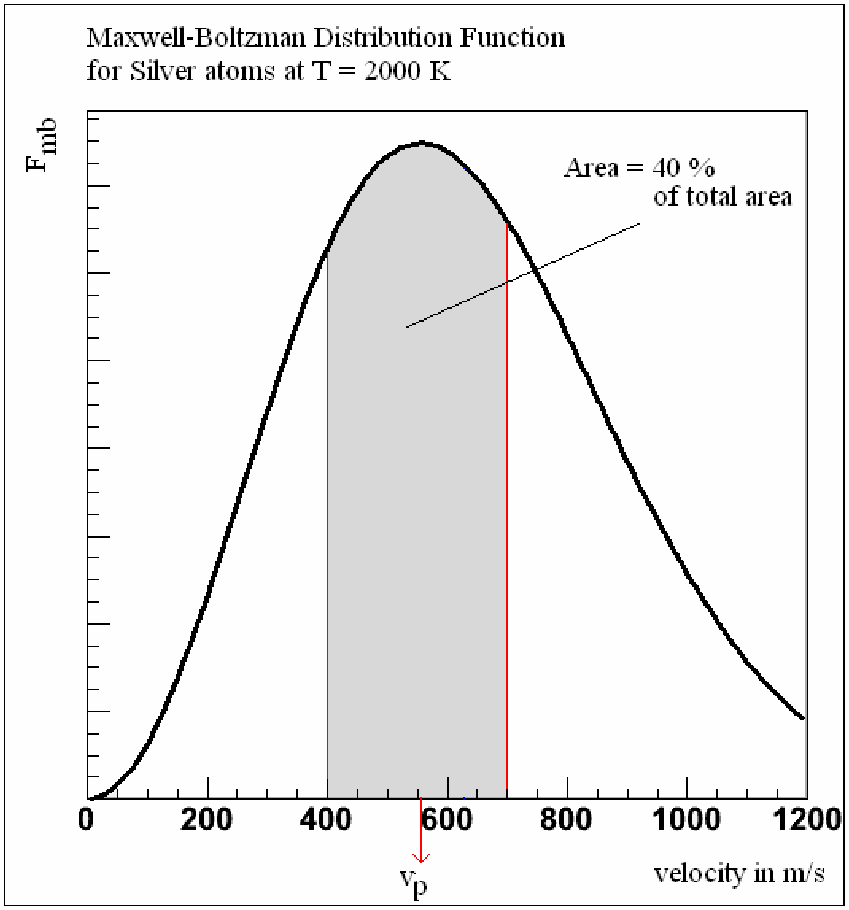
Initial velocity v of each atom is selected randomly from the Maxwell-Boltzman distribution function:

$$F_{mb} = \sqrt{2}N\pi\left(\frac{m}{\pi kT}\right)^{3/2} v^2 \exp(-\frac{mv^2}{2kT})$$

around peak value of the velocity:

$$v_p = \sqrt{2kT/m}$$



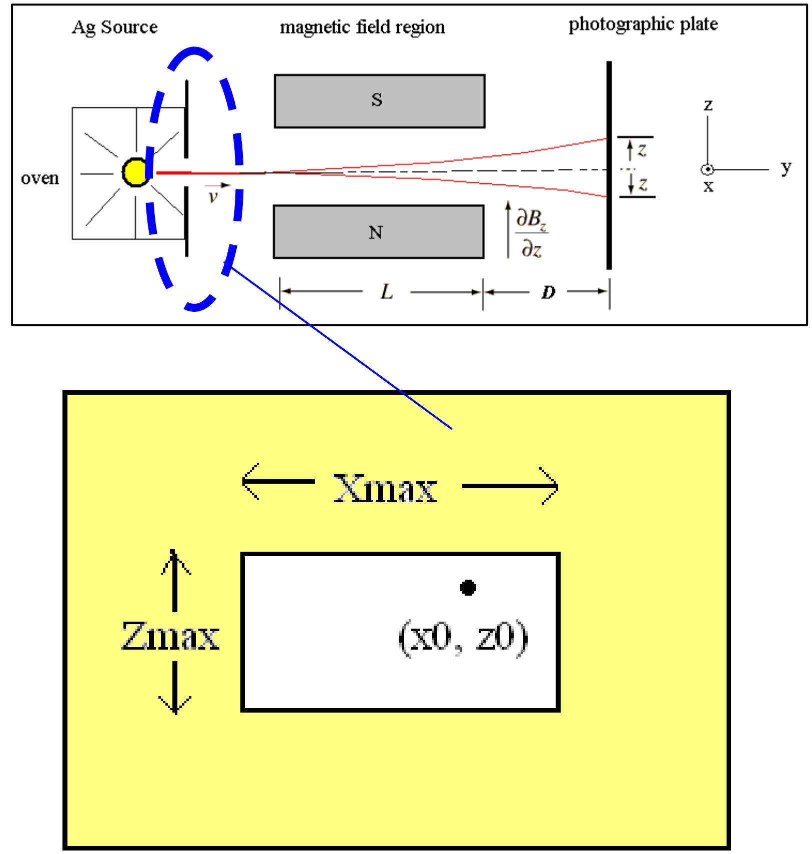


Note that:

- Components of the velocity at $(x_0, 0, z_0)$ are assumed to be: $v_{y0} = v$, and $v_{x0} = v_{z0} = 0$.
- Temperature of the oven is chosen as $T = 2000$ K.
- Mass of an Ag atom is m=1.8 x 10^{-25} kg.

Monte-Carlo Simulation

The Slit:

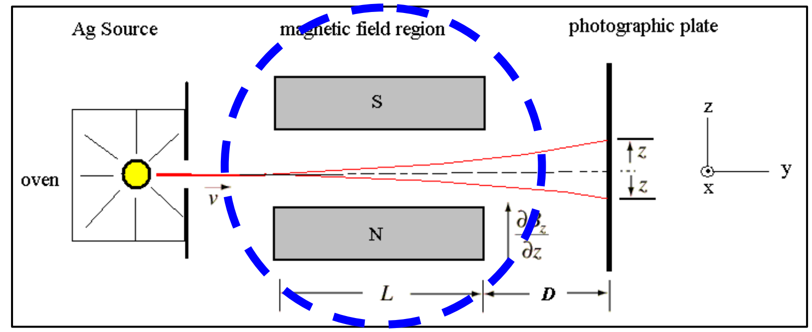


Initial position (x_0, 0, y_0), of each atom is seleled randomly from a uniform distribution.

That means: the values of x_0 and z_0 are populated randomly in the range of [Xmax, Zmax], and at that point, each atom has the velocity (0, *v*, 0).

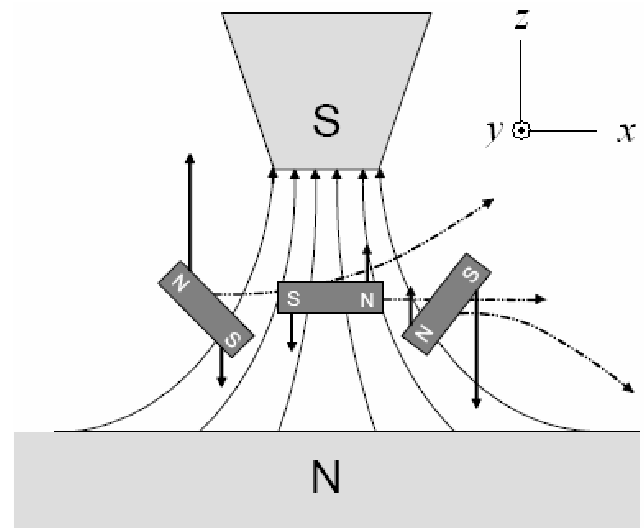
Monte-Carlo Simulation

The Magnetic Field:



In the simulation, for the field gradient (dB/dz) along z axis, we assumed the following 3-case:

- *uniform magnetic field*: $\partial B_z / \partial z = 0$
- *constant gradient* : $\partial B_z / \partial z = 100 \text{ T/m}$
- *field gradient is modulated by a Gaussian* i.e. $\partial B_z / \partial z = 100 \exp(-kx^2)$



We also assumed that along beam axis:

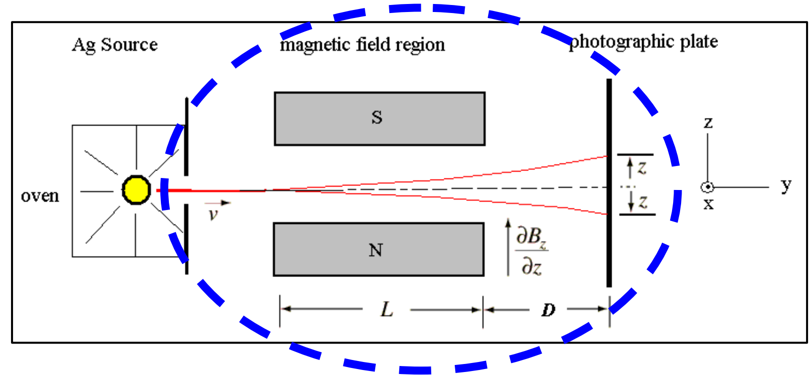
$$\partial B_z / \partial x = 0$$

$$\partial B_x / \partial z = 0 \qquad B_y = 0$$

$$\partial B_x / \partial x \approx 0$$

Monte-Carlo Simulation

Equations of motion:



Potential Energy of an electron:

$$U = -\boldsymbol{\mu}_s \cdot \mathbf{B} = -\mu_{sx} B_x - \mu_{sy} B_y - \mu_{sz} B_z$$

Componets of the force:

$$F_x = -\tfrac{\partial U}{\partial x} = \mu_{sx} \frac{\partial B_x}{\partial x} + \mu_{sz} \frac{\partial B_z}{\partial x} \approx 0 \qquad (\text{since } \partial B_x / \partial x \approx 0 \text{ and } \partial B_z / \partial x = 0)$$

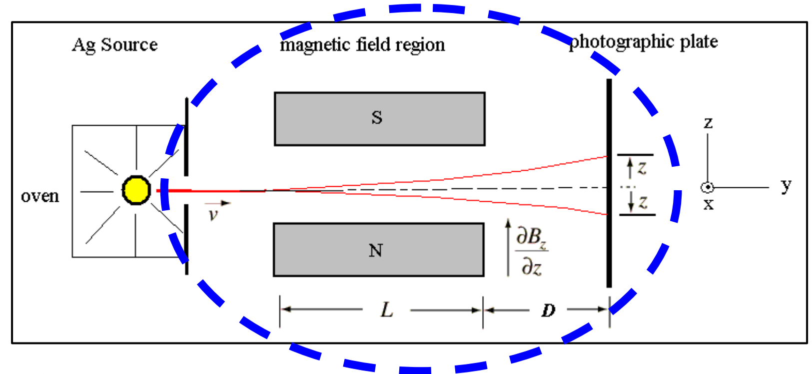
$$F_y = -\tfrac{\partial U}{\partial y} = 0 \qquad (\text{since } B_y = 0)$$

$$F_z = -\tfrac{\partial U}{\partial z} = \mu_{sx} \frac{\partial B_z}{\partial z} + \mu_{sz} \frac{\partial B_z}{\partial z} = \mu_{sz} \frac{\partial B_z}{\partial z} \qquad (\text{since } \partial B_z / \partial z = 0)$$

Consequently we have,

$$\boxed{F_x \approx 0} \qquad \boxed{F_y = 0} \qquad \boxed{F_z = \mu_{sz} \frac{\partial B_z}{\partial z} = \mu_s \cos\theta \frac{\partial B_z}{\partial z}}$$

Monte-Carlo Simulation



Equations of motion:

Differential equations and their solutions:

$$a_x = \frac{d^2x}{dt^2} = \frac{F_x}{m_{Ag}} \approx 0$$

$$x = x_0 + v_{0x}t$$

since $v_{0x} = 0$

$$x = x_0$$

$$a_y = \frac{d^2y}{dt^2} = \frac{F_y}{m_{Ag}} = 0$$

$$y = y_0 + v_y t$$

since $v_{0y} = v$ and $y_0 = 0$

$$y = vt$$

$$a_z = \frac{d^2z}{dt^2} = \frac{F_z}{m_{Ag}} = \frac{\mu_{sz}\partial B_z / \partial z}{m_{Ag}}$$

$$z = z_0 + v_{0z}t + \frac{1}{2}a_z t^2$$

since $v_{0z} = 0$

$$z = z_0 + \frac{1}{2}a_z t^2$$

So the final positions on the photographic plate in terms of *v*, *L and D*:

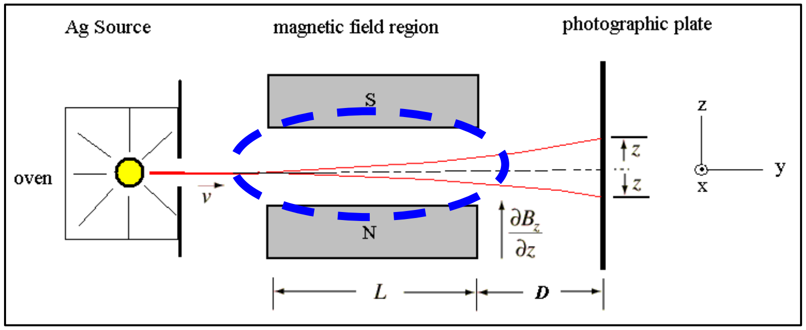
$$\boxed{x = x_0}$$

$$\boxed{y = L + D}$$

$$\boxed{z = z_0 + \frac{1}{2}a_z\left(\frac{L}{v}\right)^2 + D\sqrt{\frac{2a_z L}{v}}}$$

Here x_0 and z_0 are the initial positions at $y = 0$.

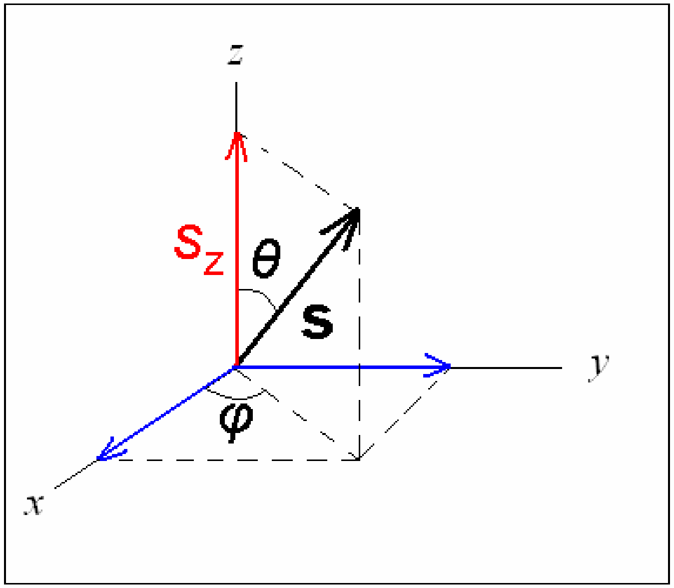
Monte-Carlo Simulation



Quantum Effect:

Spin vector components:

$$\mathbf{S} = (S_x, S_y, S_z)$$



In spherical coordinates:

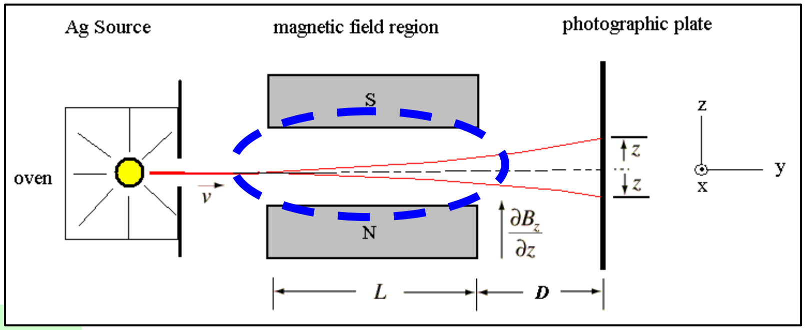
$$S_x = |\mathbf{S}| \sin(\theta) \cos(\varphi)$$

$$S_y = |\mathbf{S}| \sin(\theta) \sin(\varphi)$$

$$S_z = |\mathbf{S}| \cos(\theta)$$

where the magnitude of the spin vector is: $|\mathbf{S}| = \frac{\sqrt{3}}{2}\hbar$

Monte-Carlo Simulation

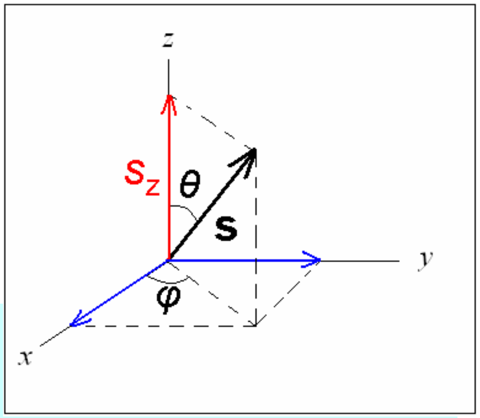


***Quantum Effect*:**

Angle φ can be selected as:

$$\varphi = 2\pi R$$

where R is random number in the range (0,1).



However, angle θ can be selected as follows:

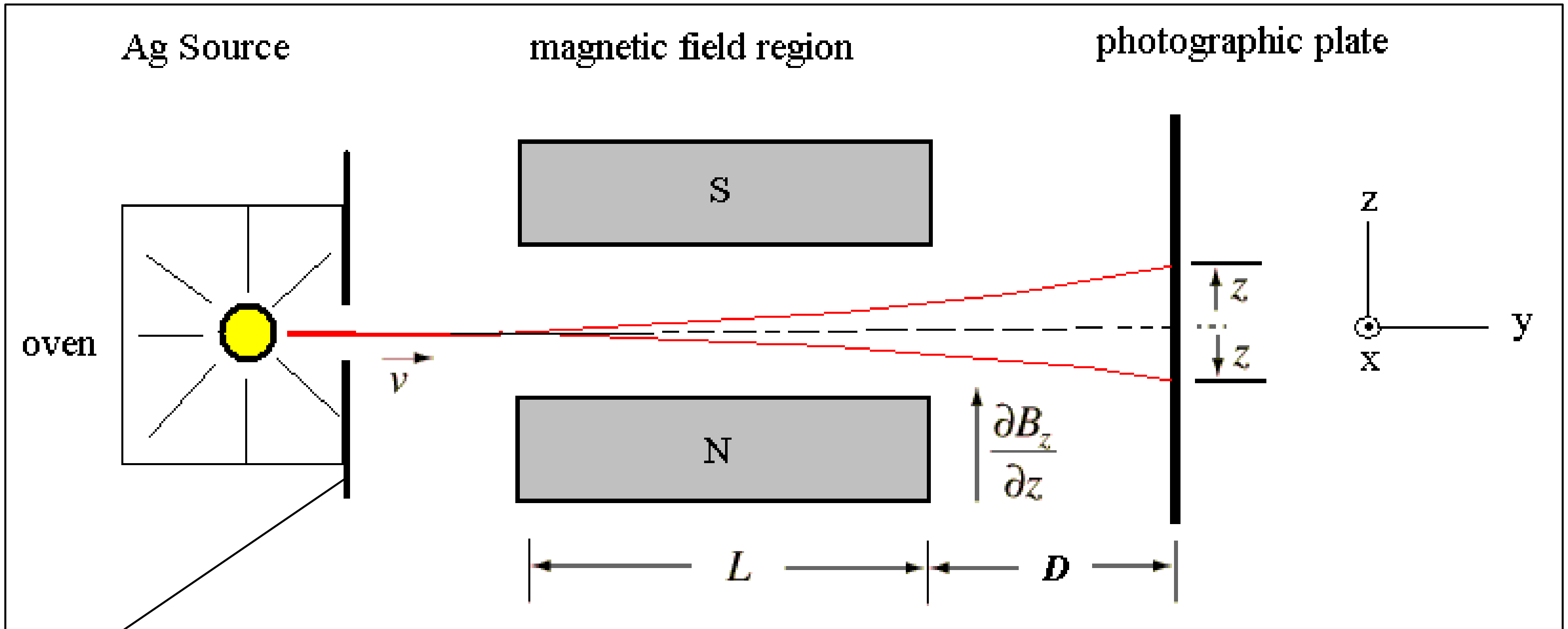
- *if S_z is **<u>not</u>** quantized, cosθ will have uniform random values:*

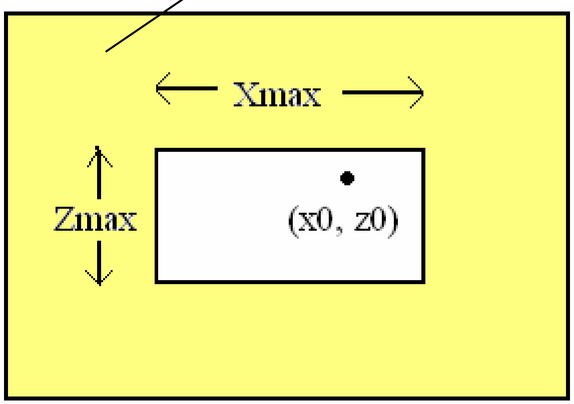
$$\cos\theta = 2R - 1$$

- *else if S_z is quantized, cosθ will have only two random values:*

$$\cos\theta = \frac{S_z}{S} = \frac{\pm\hbar/2}{\sqrt{3}\hbar/2} = \pm\frac{1}{\sqrt{3}}$$

Monte-Carlo Simulation





Geometric assumptions in the simulation:

- L = 100 cm and D = 10 cm
- Xmax = 5 cm and Zmax = 0.5 cm

Monte-Carlo Simulation

Physical assumptions in the simulation:

- N = 10,000 or N = 100,000 Ag atoms are selected.
- Velocity (v) of the Ag atoms is selected from Maxwell–Boltzman distribution function around peak velocity.
- The temperature of the Ag source is takes as T = 2000 K.
  (For the silver atom: Melting point T = 1235 K ; Boiling point 2435 K)

- Field gradient along z axis is assumed to be:
  - $\partial B_z / \partial z = 0$ *uniform magnetic field*
  - $\partial B_z / \partial z = 100 \ \mathrm{T/m}$ *constant field gradient along z axis*
  - $\partial B_z / \partial z = 100 \ \exp(-kx^2)$ *field gradient is modulated by a Gaussian*

- z component of the spin (S_z) is
  - either quantized according to quantum theory such that cosθ = 1/sqrt(3)
  - or cosθ is not quantized and assumed that it has random orientation.

Results

- Hereafter slides, you will see some examples of simulated distributions that are observed on the photographic plate.

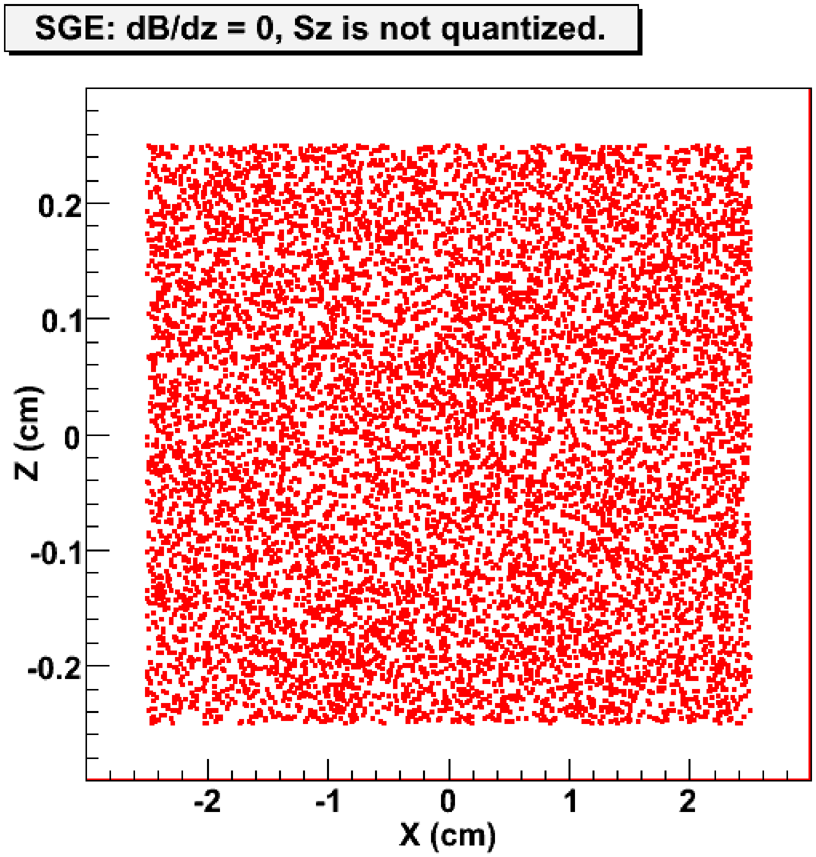
- Each red point represents a single Ag atom.

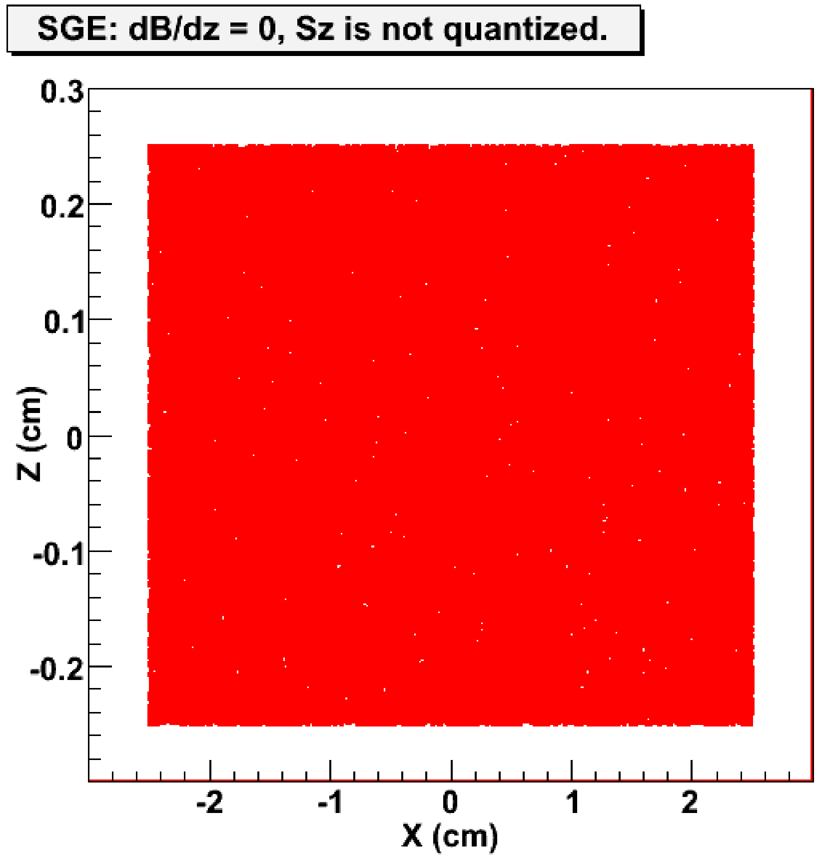
- You can find the source codes of the simulation implemented in Fortran 90, ANSI C and ROOT programming languages at:

  http://www1.gantep.edu.tr/~bingul/seminar/spin

Results **dB/dz = 0**

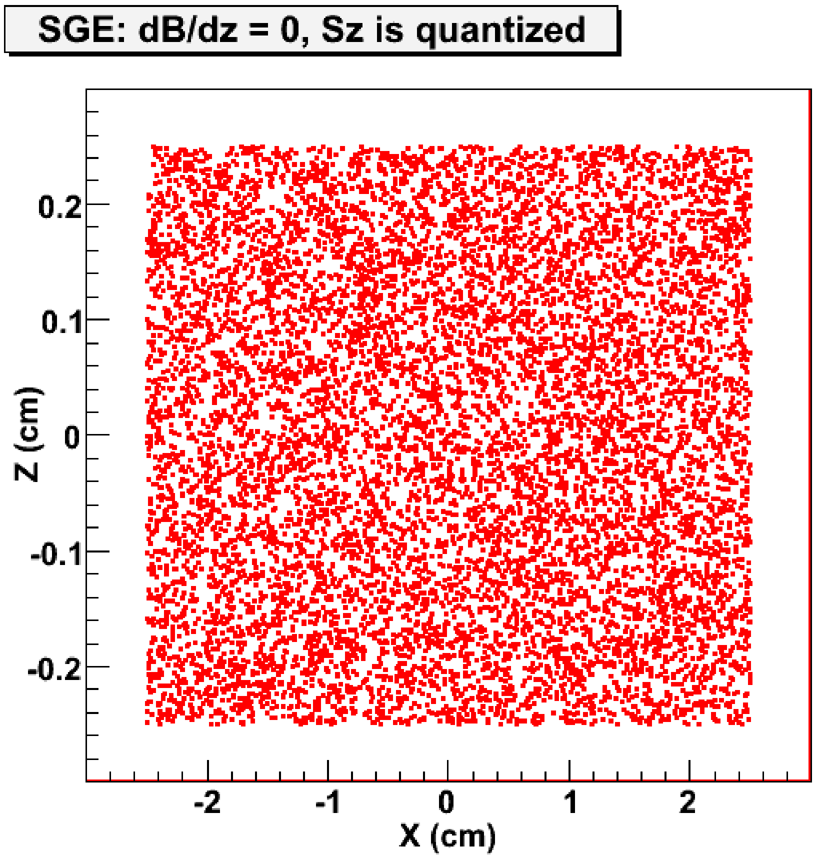
$N = 10{,}000$ $N = 100{,}000$



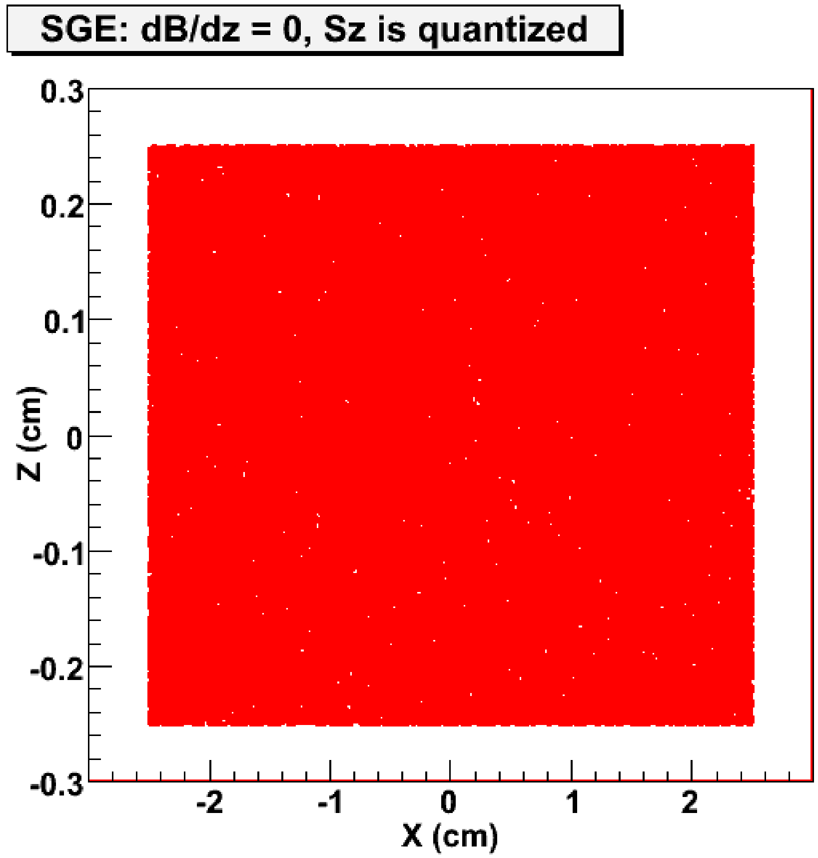


Results   dB/dz = 0

$N = 10{,}000$

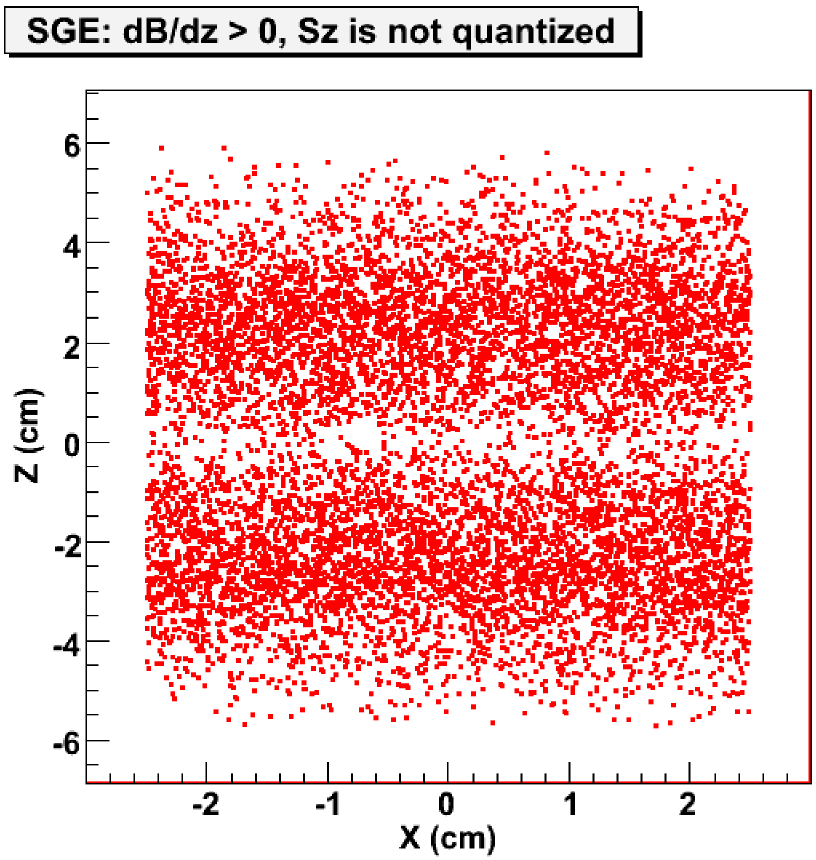


$N = 100{,}000$

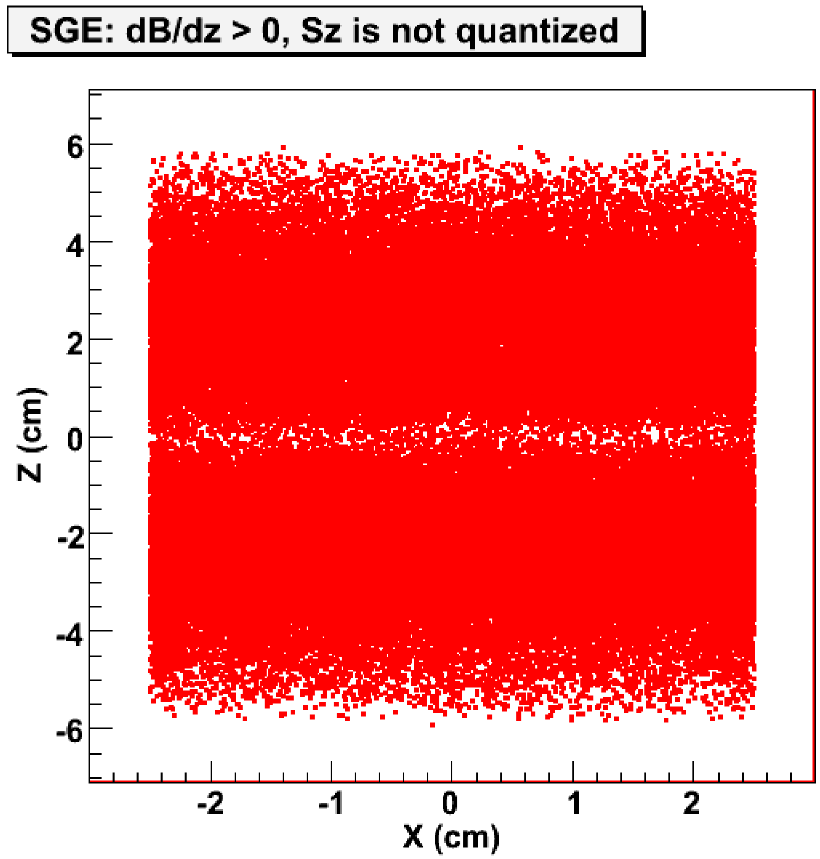


Results   dB/dz = constant > 0

$N = 10,000$

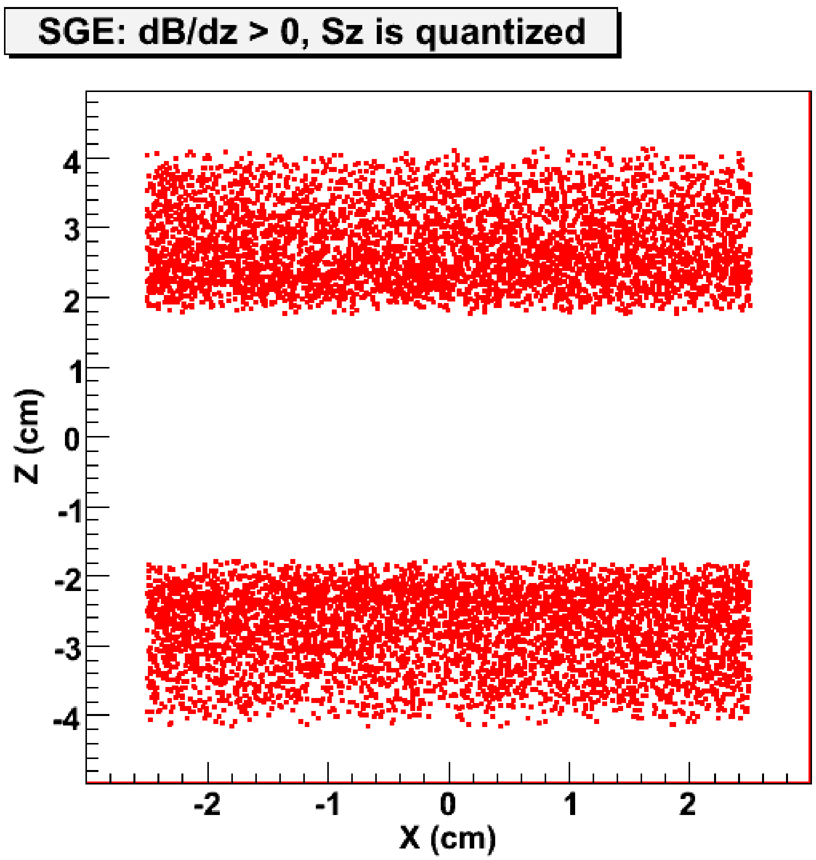


$N = 100,000$

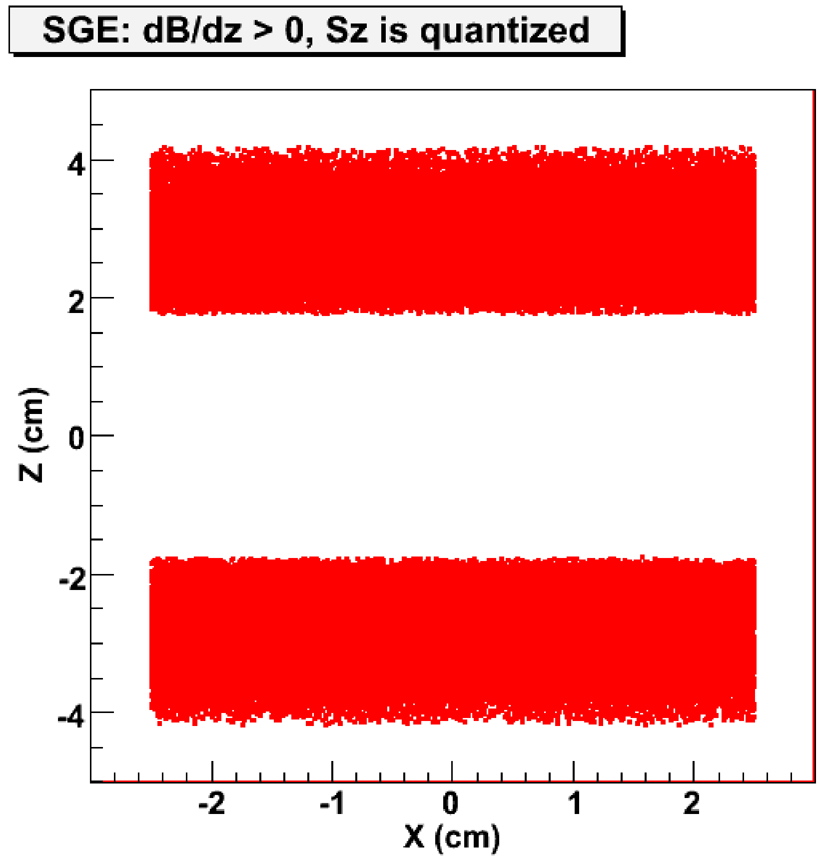


Results dB/dz = constant > 0

$N = 10,000$



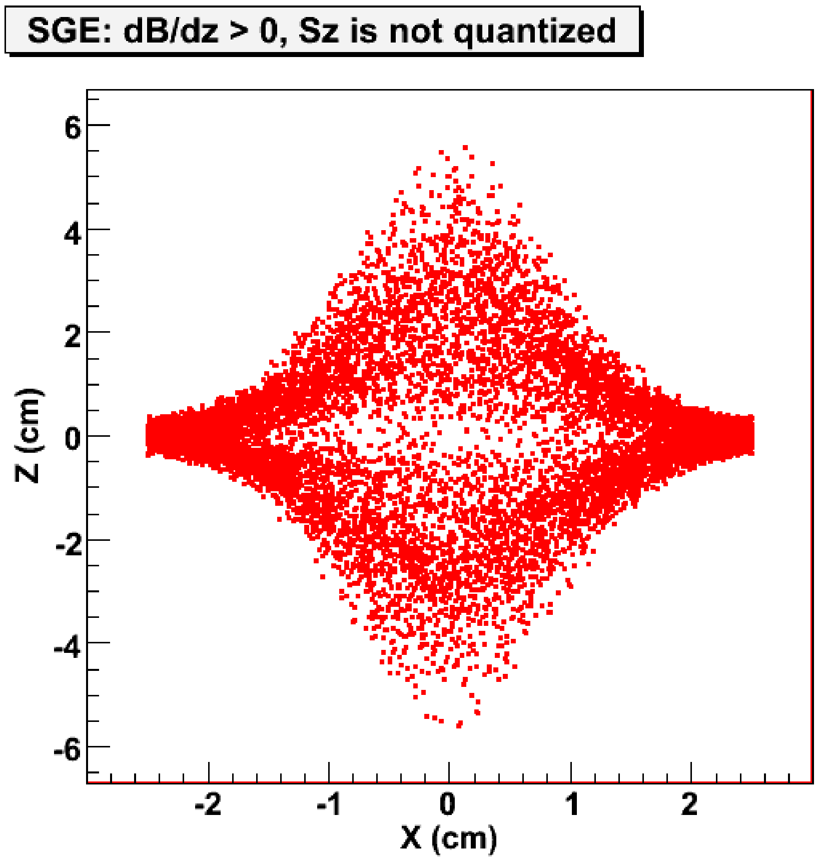
$N = 100,000$



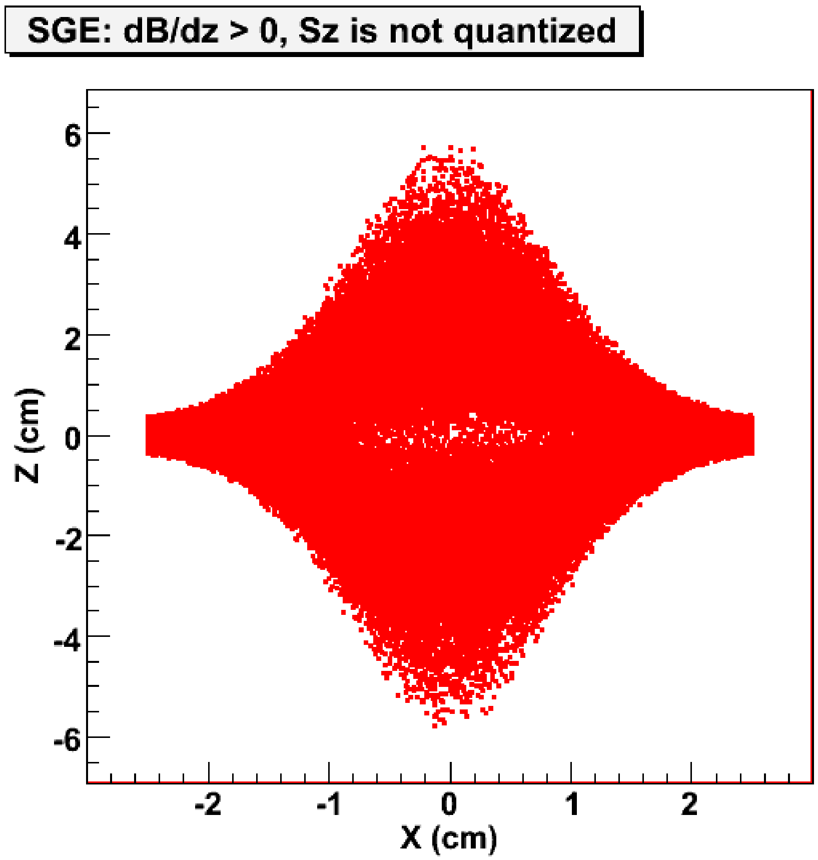
Results

dB/dz = constant * exp(−kx²)

N = 10,000



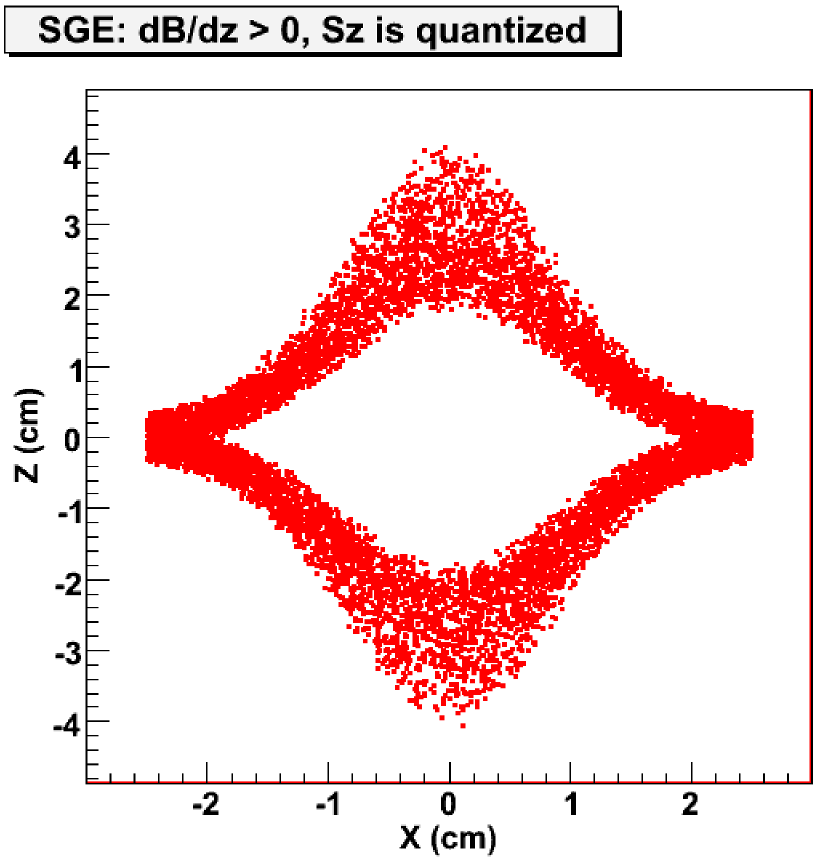
N = 100,000



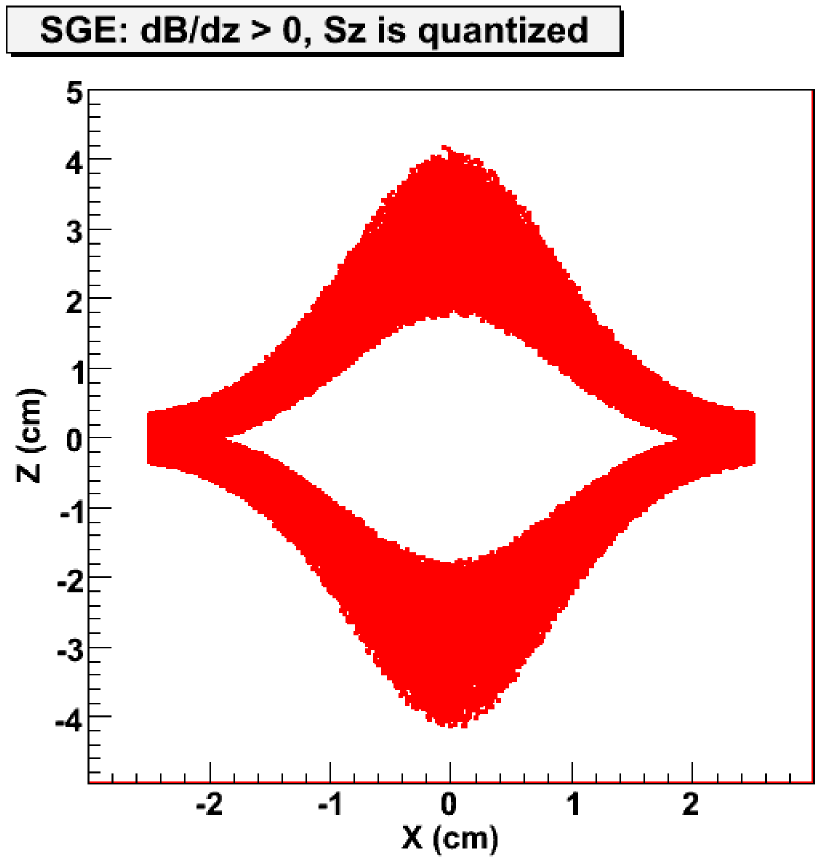
Results

dB/dz = constant * exp(–kx^2)

N = 10,000



N = 100,000



End of Seminar

Thanks.

April 2008