5.7 Improved Compressibility Corrections

Linearized solutions are influenced predominantly by free-stream conditions; they do not fully recognize changes in local regions of the flow. Laitone applied (219) locally in the flow: $\longrightarrow$ $C_p = \dfrac{C_{p_0}}{\sqrt{1-M^2}}$ where M is the local Mach number

In turn, M can be related to M_∞ and the pressure coefficient through the isentropic flow relations. The resulting compressibility correction is:

$$C_p = \frac{C_{p_0}}{\sqrt{1-M_\infty^2} + \underbrace{\left[M_\infty^2\left(1+\frac{\gamma-1}{2}M_\infty^2\right)/2\sqrt{1-M_\infty^2}\right]C_{p_0}}} \quad \text{...(221)}$$

Note that, as C_{p_0} becomes small, Eqn (221) reaches the Prandtl-Glauert rule.

Another approach is called Karman-Tsien rule with a simplified "tangent gas" equation of state:

$$C_p = \frac{C_{p_0}}{\sqrt{1-M_\infty^2} + \dfrac{M_\infty^2}{(1+\sqrt{1-M_\infty^2})} \cdot \dfrac{C_{p_0}}{2}} \quad \text{...(222)}$$

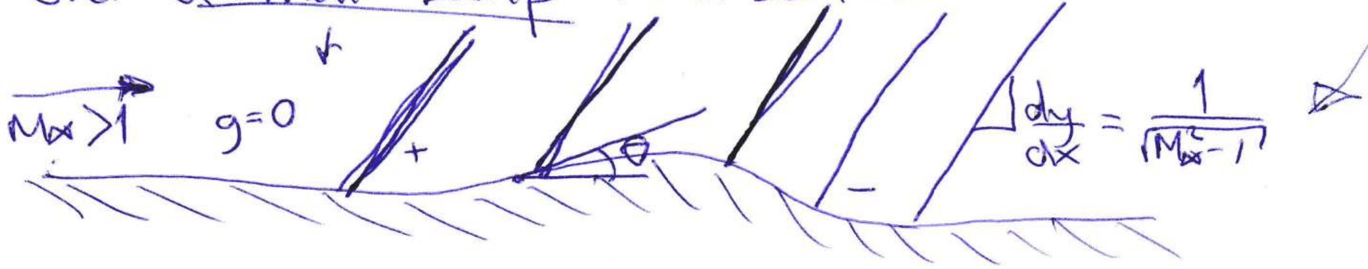
5.8 Linearized Supersonic flow

from eqn (197), $\rightarrow (1-M_\infty^2)\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = 0$, the linearized perturbation velocity potential equation for 2D flow becomes: $\underbrace{\beta^2\phi_{xx} + \phi_{yy} = 0}_{(223)}$ for subsonic flow

where $\beta = \sqrt{1-M_\infty^2}$

and the form of $\lambda^2 \phi_{xx} - \phi_{yy} = 0$ ...(224) for supersonic flow where $\sqrt{M_\infty^2 - 1}$ is λ. The difference between (223) and (224) is fundamental, for they are elliptic and hyperbolic PDEs, respectively.

Consider the supersonic flow over a body or surface which introduces small changes in the flowfield, i.e. flow over a thin airfoil, over a mildly wavy wall, or over a small bump in a surface.



Its general solution is: $\phi = f(x - \lambda y) + g(x + \lambda y)$ ...(225) which can be verified by direct substitution into (224). Examining the particular solution where $g = 0$, hence $\phi = f(x - \lambda y)$, we see that lines of constant ϕ correspond to $x - \lambda y = \text{Const}$ or $\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}}$ ...(226)

Letting $g = 0$, we have:

$$\phi = f(x - \lambda y) \rightarrow u = \frac{\partial \phi}{\partial x} = f' \quad ...(227)$$

$$v = \frac{\partial \phi}{\partial y} = -\lambda f' \quad ...(228)$$

Combining (227) & (228): $u' = -\frac{v'}{\lambda}$ ... (229)

Boundary condition is: $\tan\theta = \frac{dy}{dx} = \frac{v'}{V_\infty + u'}$ ... (230)

For small perturbations $u' \ll V_\infty$ and $\tan\theta \approx \theta$

↳ $v' = V_\infty \theta$ ... (231)

Substituting (231) into (229) → $u' = -\frac{V_\infty \theta}{\lambda}$ ... (232)

Therefore from Eqn 208, $C_p = -2u'/V_\infty$,

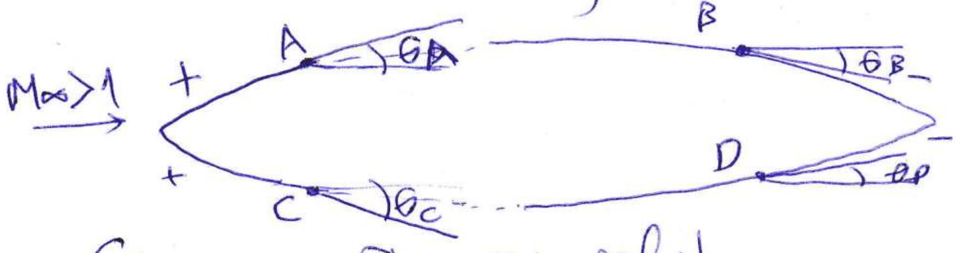
↳ $C_p = -\frac{2u'}{V_\infty} = \frac{2\theta}{\lambda}$ ... (233)

or

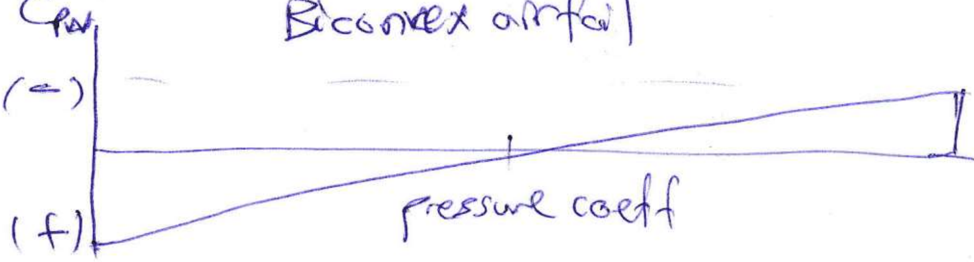
$$\boxed{C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}} \quad \text{... (234)}$$

This equation is an important result. It is the linearized supersonic surface pressure coefficient.

It states that C_p is directly proportional to the local surface inclination with respect to free-stream. It holds for any slender 2D shape. For example,



Biconvex airfoil



$C_{P_A} = \frac{2\theta_A}{\sqrt{M_\infty^2 - 1}}$, $\theta_A > 0$

$C_{P_B} = \frac{2\theta_B}{\sqrt{M_\infty^2 - 1}}$, $\theta_B < 0$

↓

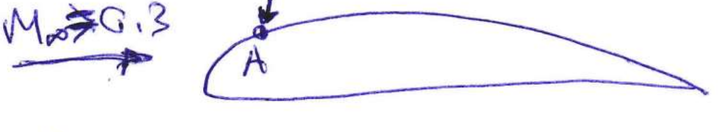
$$\underbrace{C_p = \frac{-2\theta}{M_\infty^2 - 1}}_{(235)}$$

Eqn (235) holds for a surface generating right-running waves, i.e. the bottom surfaces in figures. In both (235) & (234), θ is measured positive above the local flow direction and negative below the local flow direction. Hence on bottom surface of biconvex airfoil, θ_c is negative, θ_p is positive. In conjuction with (235), this still yields a positive C_p on the forward compression surface and a negative C_p on the rearward expansion surface.

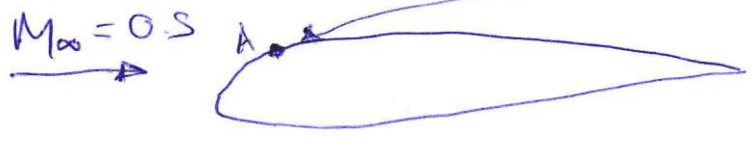
Check Ex. 9.2 from Modern Comp Flow Book

5.9 Critical Mach Number

Consider an airfoil at low subsonic speed with a free-stream Mach number $M_\infty = 0.3$. The flow expands around the top surface of the airfoil, dropping to a minimum pressure at point A. At this point, the local Mach number on the surface will be a maximum, in this case $M_A = 0.435$.

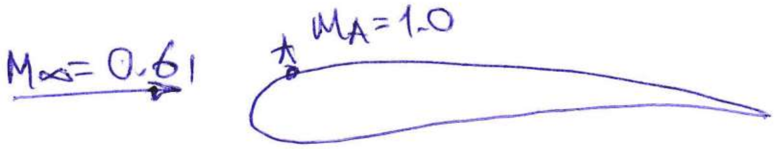


Now assume that we increase M_∞ to 0.5. The local Mach number at the minimum pressure point is 0.772.



Now, lets increase M_∞ to just the right value such that $M_A = 1.0$ at the minimum pressure point. This value is $M_\infty = 0.61$. When this occurs, M_∞ is called the

critical Mach number



By definition, the critical Mach number is that free-stream Mach number at which sonic flow is first encountered on the airfoil.

The critical Mach number can be calculated as follows:

- Assuming isentropic flow throughout the flow field

$$\frac{P_A}{P_\infty} = \left(\frac{1 + \frac{\gamma - 1}{2} M_\infty^2}{1 + \frac{\gamma - 1}{2} M_A^2} \right)^{\gamma/(\gamma-1)} \quad \text{--- (236)}$$

- Combining eqns (201), $C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{P}{P_\infty} - 1\right)$ with (236):

$$C_{P_A} = \frac{2}{\gamma M_\infty^2} \left[\left(\frac{1 + \frac{\gamma - 1}{2} M_\infty^2}{1 + \frac{\gamma - 1}{2} M_A^2} \right) - 1 \right] \quad \text{--- (237)}$$

Now assume as before that point A is the minimum pressure (hence maximum-velocity) point on the airfoil. Furthermore, assume $M_A = 1$. Then by definition, $M_\infty \equiv M_{cr}$. Also, $C_p \equiv C_{p_{cr}}$, Setting $M_A = 1$, $M_\infty = M_{cr}$ and $C_p \equiv C_{p_{cr}}$:

$$C_{p_{cr}} = \frac{2}{\gamma M_{cr}^2} \cdot \left[\left(\frac{1 + \frac{\gamma - 1}{2} M_{cr}^2}{1 + \frac{\gamma - 1}{2}} \right)^{\gamma} - 1 \right] \quad \text{---} (238)$$

↳ So, $C_{p_{cr}} = f(M_{cr})$

You can calculate the critical Mach number for a given airfoil using these steps:

1- Obtain as given data a measured or calculated value of the incompressible pressure coefficient at the minimum pressure point, C_{p_0}.

2- Using one of the compressibility corrections, plot C_p as a function of M_∞

3- Using Eqn (238) plot $C_{p_{cr}}$ as a function of M_{cr}.

4- The intersection of curves B and C defines the critical Mach number for the given airfoil.