ME209 - Numerical Methods

Lecture 3: Roots of Equations

By:

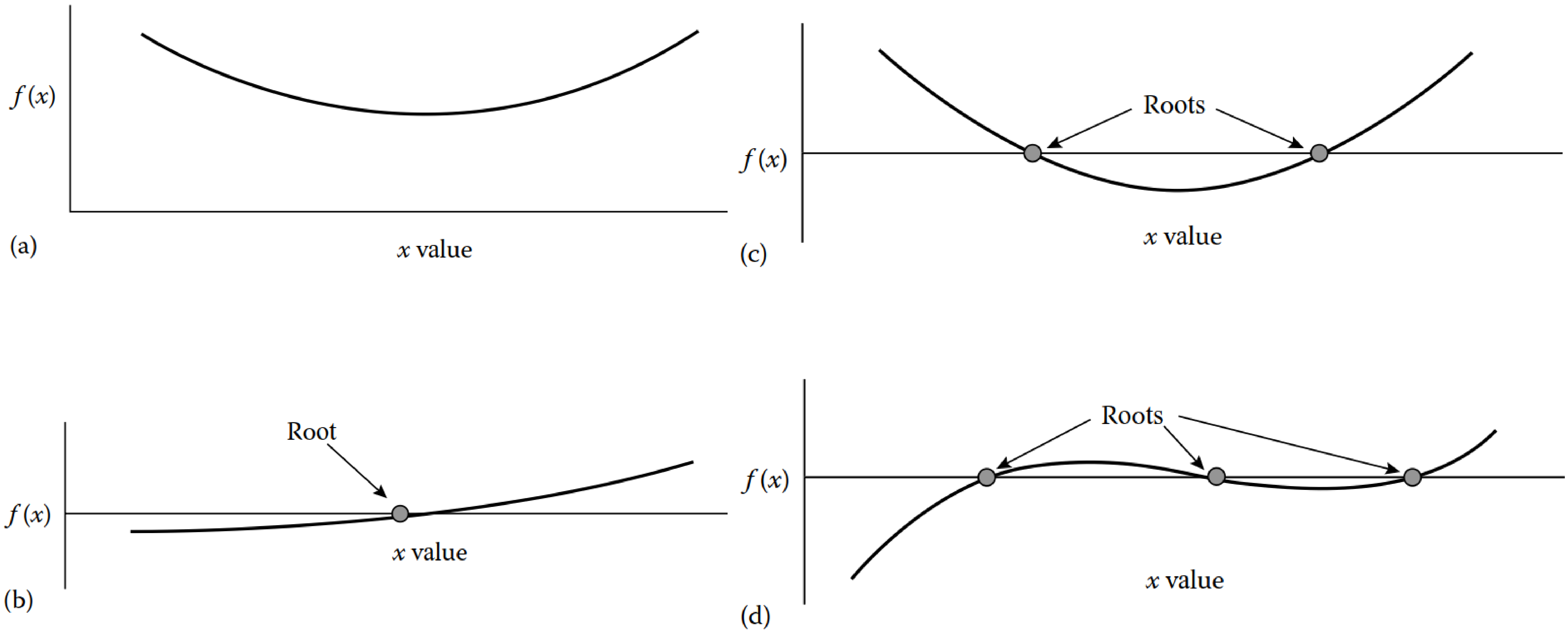
Prof. Dr. Ahmet Erkliğ

Introduction

Any nonlinear equation can, after suitable algebraic manipulation, be expressed as a function of the following form:

$$f(x) = 0$$

The values of *x* that satisfy the condition of equation are termed the roots of the function; these roots are the solution to the equation $f(x) = 0$



Roots for selected functions: (a) no roots, (b) one root, (c) two roots, and (d) three roots.

Introduction

For very simple functions, the roots can often be found analytically· For example, for the general second-order polynomial $ax^2 + bx + c = 0$, the roots x_1 and x_2 can be found analytically using the following equation:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The roots for a third-order polynomial can also be found analytically· However, a general solution does not exist for other higher order polynomials·
It is more difficult to obtain analytical solutions for the roots of non-polynomial, nonlinear functions·

Introduction

- However, there are many complex equations whose roots we cannot easily find analytically. In such cases, numerical methods can be used as powerful alternatives in finding the roots of equations.

- Numerical methods commonly used in finding the roots of equations can be examined in two groups. These are: ***Bracketing methods*** and ***open methods***.

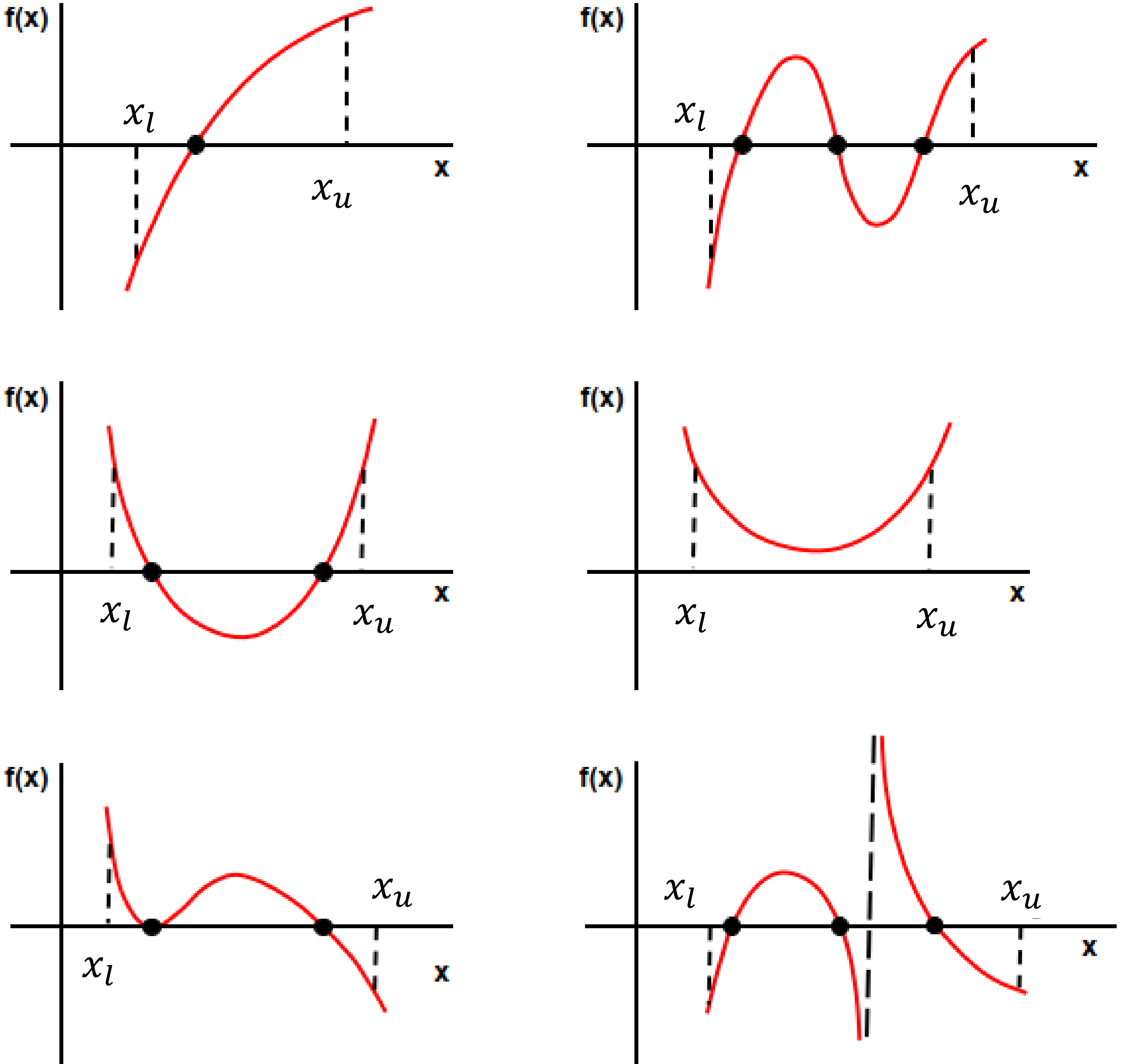
Bracketing methods:

- Bisection Method
- False-Position Method

Open methods:

- One-point Iteration Method
- Newton-Raphson Method
- Secant Method

General Idea of Bracketing Methods



RULE 1: IF $f(x_l) * f(x_u) < 0$ THEN

there are odd number of roots.

RULE 2: IF $f(x_l) * f(x_u) > 0$ THEN

there are: (i) even number of roots,

(ii) no roots.

VIOLATIONS:

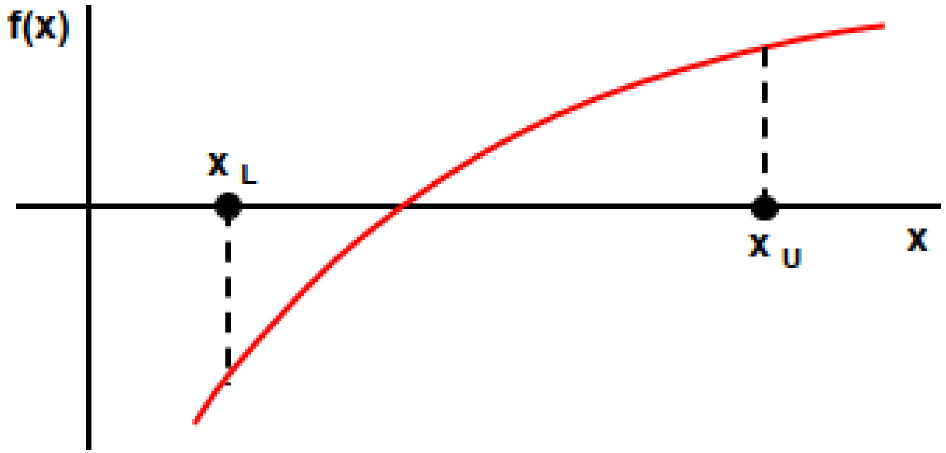
(i) multiple roots,

(ii) discontinuities.

Bisection Method

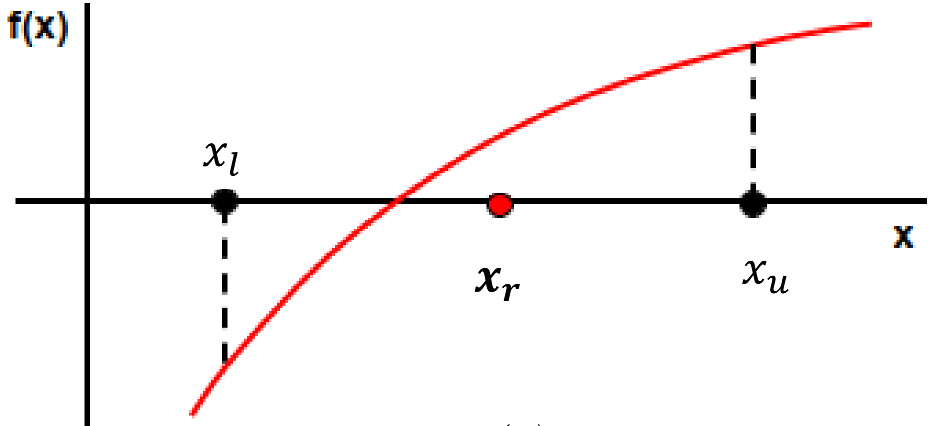
- The bisection method is an extension of the direct-search method for cases when it is known that only one root occurs within a given interval of x. For the same level of precision, the bisection method will, in general, require fewer calculations than the direct-search method.
- The bisection method is one type of incremental search method in which the interval is always divided in half.
- If a function changes sign over an interval, the function value at the midpoint is evaluated.
- The location of the root is then determined as lying at the midpoint of the subinterval within which the sign change occurs.
- The process is repeated to obtain refined estimates.
- The bisection method will always converge on the root, provided that only one root lies within the starting interval for x·

Bisection Method

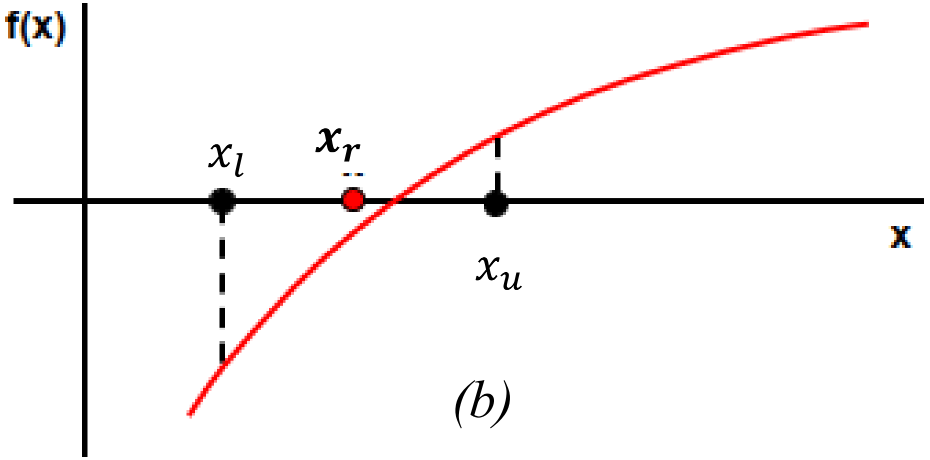


- **Step 1:** Choose two initial estimations, x_{LOWER} (x_l) and x_{UPPER} (x_u).
- They should bracket the root, i.e.

$$f(x_l) * f(x_u) < 0$$



(a)



(b)

Step 2: Estimate the root as a midpoint of this interval

$$\boldsymbol{x_r = \frac{(x_l + x_u)}{2}}$$

Step 3: Determine the interval which contains the root,

(a) IF $f(x_l) * f(x_r) < 0$ THEN the root is between x_L and x_r, Therefore, set $x_u = x_r$ and RETURN to step 2.

(b) IF $f(x_l) * f(x_r) > 0$ THEN the root is between x_r and x_U. Therefore, set $x_l = x_r$ and RETURN to step 2.

(c) IF $f(x_l) * f(x_r) = 0$, the root is equals x_r; TERMINATE the computation

Termination Criteria and Error Estimates

We need a termination criteria to end the iteration. Here, an approximate percent relative error ε_a can be calculated as:

$$\boxed{\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| 100\%}$$

When the approximate percent relative error ε_a falls below the specified percent relative error ε_s or tolerance, we could terminate the bisection method.

To terminate calculations $\varepsilon_a < \varepsilon_s$

Example 3.1: Find the square root of 11. (Tolerance value: $|\varepsilon_s| = 0.5\%$)

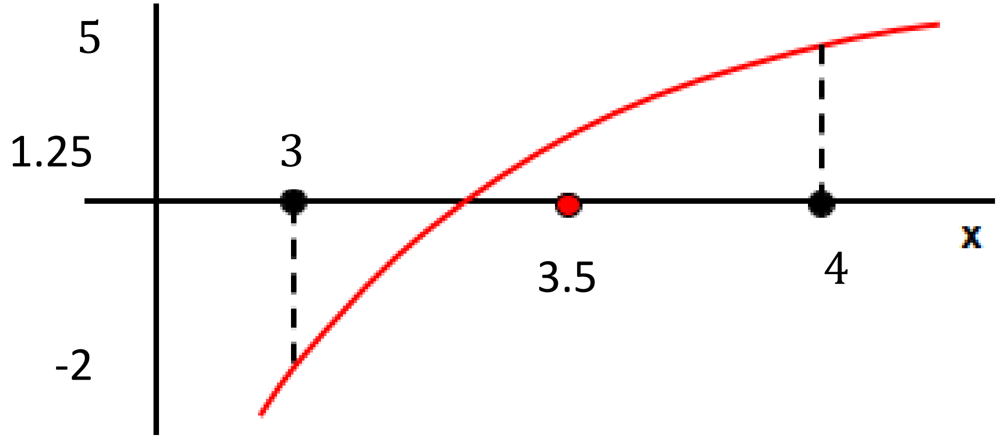
$$x^2 = 11 \;\rightarrow f(x) = x^2 - 11 \quad (exact\ solution\ is\ 3.31662479)$$

Choose initial estimates: $3^2 = 9 < 11$ and $4^2 = 16 > 11 \;\rightarrow \boldsymbol{x_l} = \mathbf{3}$ and $\boldsymbol{x_u} = \mathbf{4}$

1st iteration: $x_r = \frac{x_l + x_u}{2} = \frac{3+4}{2} \rightarrow x_r = 3.5$ and $f(3.5) = 3.5^2 - 11 = 1.25$

$f(x_l) * f(x_r) = -2 * 1.25 \rightarrow -2.5 < 0$ (root is between x_l and x_r)

SET $x_u = x_r$=3.5 and CONTINUE to iterate.

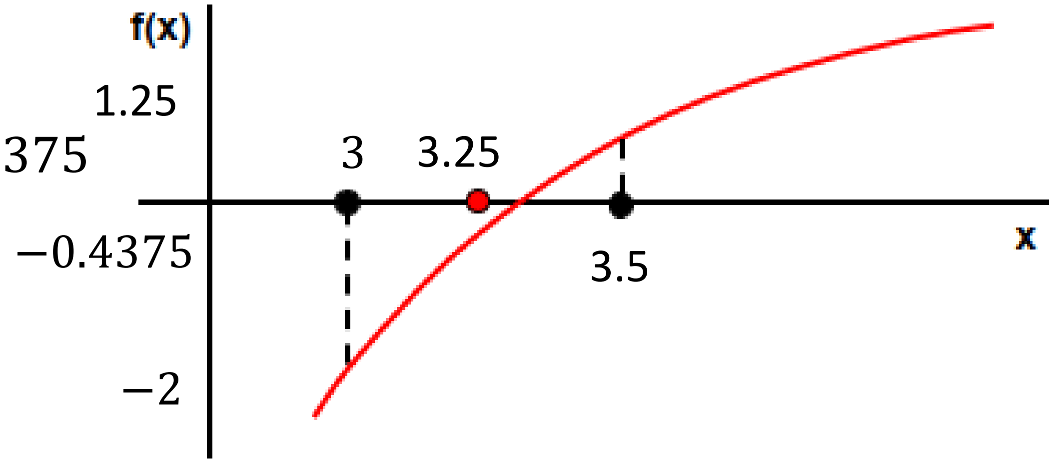


2nd iteration: $\boldsymbol{x_l = 3, x_u = 3.5}$

$$x_r = \frac{3+3.5}{2} \rightarrow x_r = 3.25 \text{ and } f(3.25) = 3.25^2 - 11 = -0.4375$$

$$f(x_l) * f(x_r) = -2 * -0.4375 = 0.875 > 0$$
(root is between x_r and x_u)

SET $x_l = x_r = 3.25$ and CONTINUE to iterate.

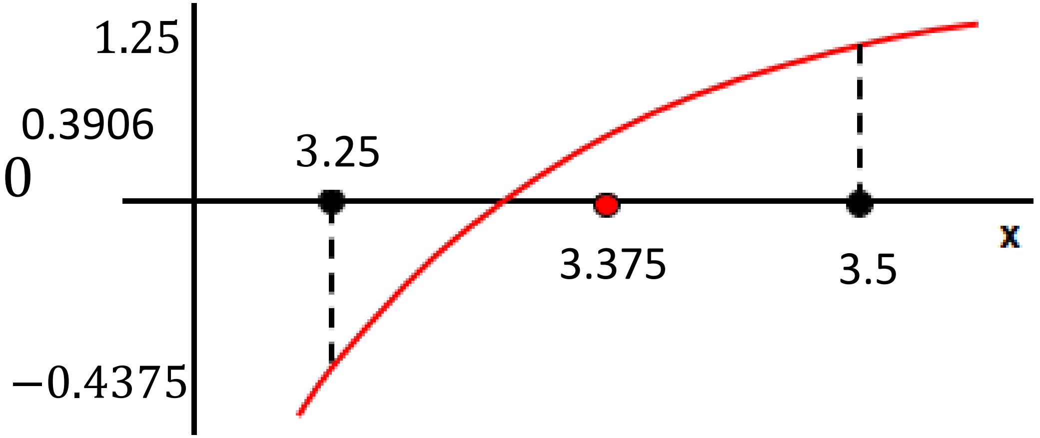


3rd iteration: $\boldsymbol{x_l = 3.25, x_u = 3.5}$

$$x_r = \frac{3.25+3.5}{2} \rightarrow x_r = 3.375 \text{ and } f(3.375) = 0.390625$$

$$f(x_l) * f(x_r) = -0.4375 * 0.390625 = -0.17089 < 0$$
(root is between x_l and x_r)

SET $x_u = x_r = 3.375$ and CONTINUE to iterate.



4th iteration: $\boldsymbol{x_l = 3.25, x_u = 3.375}$

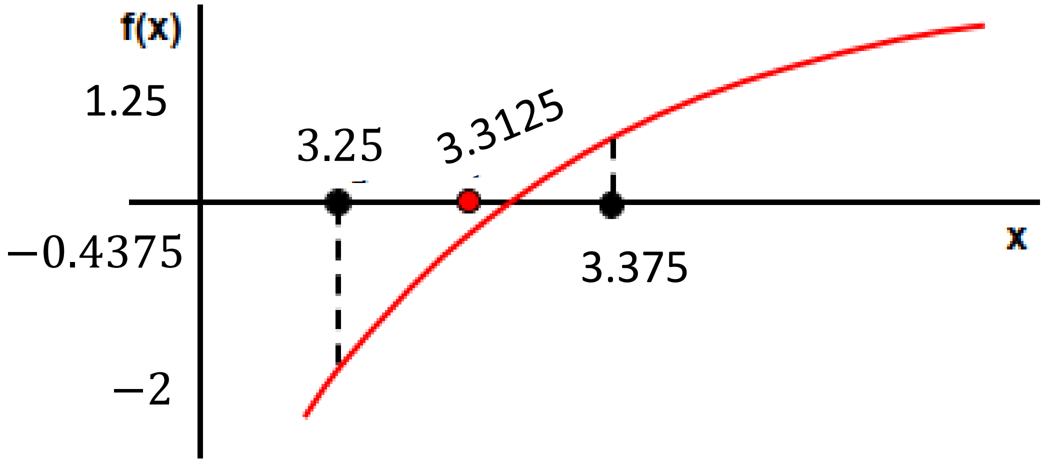
$x_r = \frac{3.25+3.375}{2} \rightarrow x_r = 3.3125$

and $f(3.3125) = 3.3125^2 - 11 = -0.027343$

$-0.4375 * -0.027343 = 0.011963 > 0$

(root is between x_r and x_u)

SET $x_l = x_r = 3.3125$ and CONTINUE to iterate.



5th iteration: $\boldsymbol{x_l = 3.3125, x_u = 3.375}$

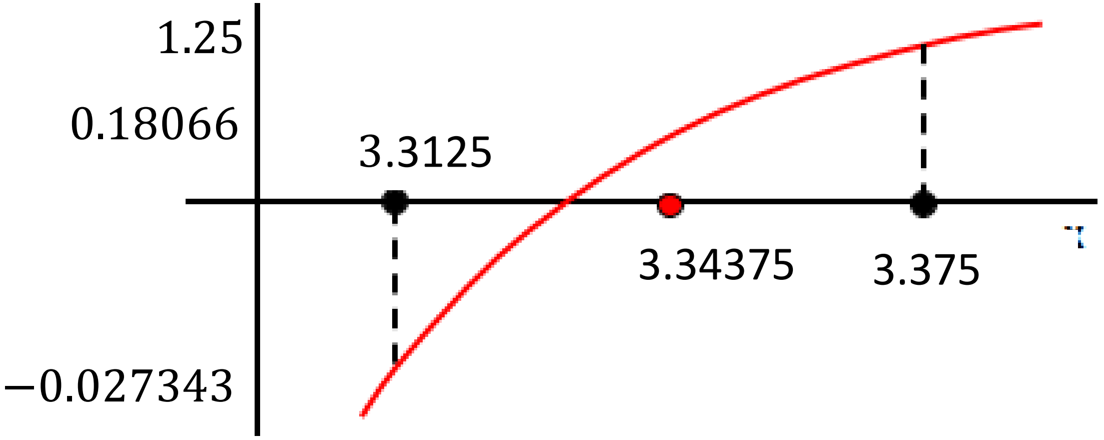
$x_r = \frac{3.3125+3.375}{2} \rightarrow x_r = 3.34375$ and

$f(3.34375) = 0.18066$

$-0.027343 * 0.390625 = -0.01068 < 0$

(root is between x_l and x_r)

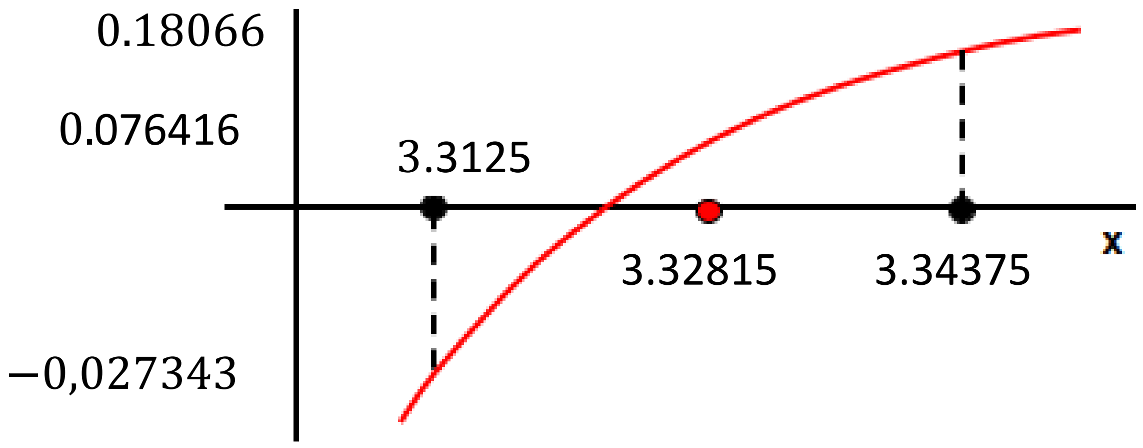
SET $x_u = x_r = 3.34375$ and CONTINUE to iterate.



6th iteration: $\boldsymbol{x_l = 3.3125, x_u = 3.34375}$

$$x_r = \frac{3.3125+3.34375}{2} \rightarrow x_r = 3.32815 \text{ and}$$
$$f(3.32815) = 0.076416$$

$$-0.027343 * 0.076416 = -0.002089 < 0$$
(root is between x_l and x_r)



Iteration	x_l	x_r	x_u	$f(x_l)$	$f(x_r)$	$f(x_l)f(x_r)$	$\lvert\varepsilon_a\rvert$ (%)
1	3	3.5	4	-2	1.25	-2.5	-
2	3	3.25	3.5	-2	-0.4375	0.875	7.69
3	3.25	3.375	3.5	-0.4375	0.390625	-0.17089	3.70
4	3.25	3.3125	3.375	-0.4375	-0.027343	0.011963	1.88
5	3.3125	3.34375	3.375	-0.027343	0.18066	-0.004942	0.93
6	3.3125	$\boxed{3.32815}$	3.34375	-0.027343	0.076582	-0.0020939	$\boxed{0.468}$

$\sqrt{11} \approx 3.32815$

$|\varepsilon_a| < |\varepsilon_s|$

Example 2

Calculate root of following polynolmial

$$f(x) = x^3 - x^2 - 10x - 8 = 0$$

Finding the roots within the interval $3.75 \leq x \leq 5.00$ to a relative accuracy as an absolute value between successive iterations of 0.01.

Check: $f(x=3.75) = -6.82$

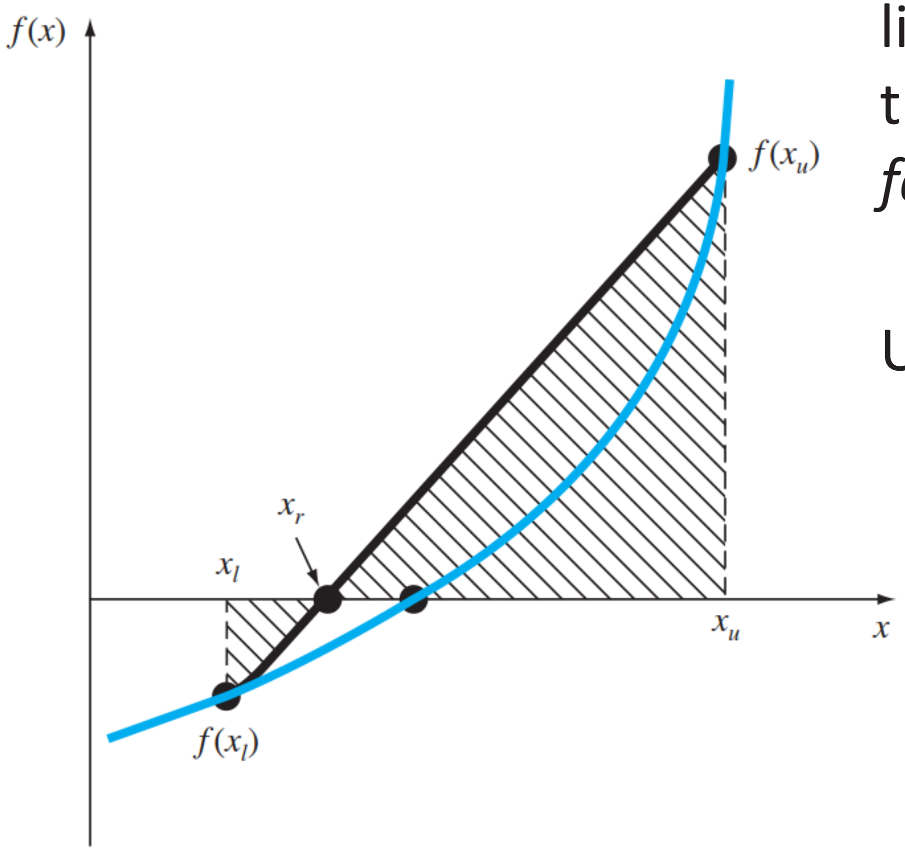
$f(x=5.00) = 42$

$fl \times fu = -286.44 < 0$

Iteration i	x_l	x_r	x_u	$f(x_l)$	$f(x_m)$	$f(x_u)$	Subinterval Containing Root $f(x_l)f(x_r)$	$f(x_r)f(x_u)$	Error ϵ
1	3.750	4.375	5.000	–6.830	12.850	42.000	–	+	–
2	3.750	4.062	4.375	–6.830	1.903	12.850	–	+	0.313
3	3.750	3.906	4.062	–6.830	–2.724	1.903	+	–	0.156
4	3.906	3.984	4.062	–2.724	–0.477	1.903	+	–	0.078
5	3.984	4.023	4.062	–0.477	0.696	1.903	–	+	0.039
6	3.984	4.004	4.023	–0.477	0.120	0.696	–	+	0.019
7	3.984	3.994	4.004	–0.477	–0.180	0.120	+	–	0.010

False-Position Method

An alternative method that exploits this graphical insight is to join $f(x_l)$ and $f(x_u)$ by a straight line. The intersection of this line with the x axis represents an improved estimate of the root.

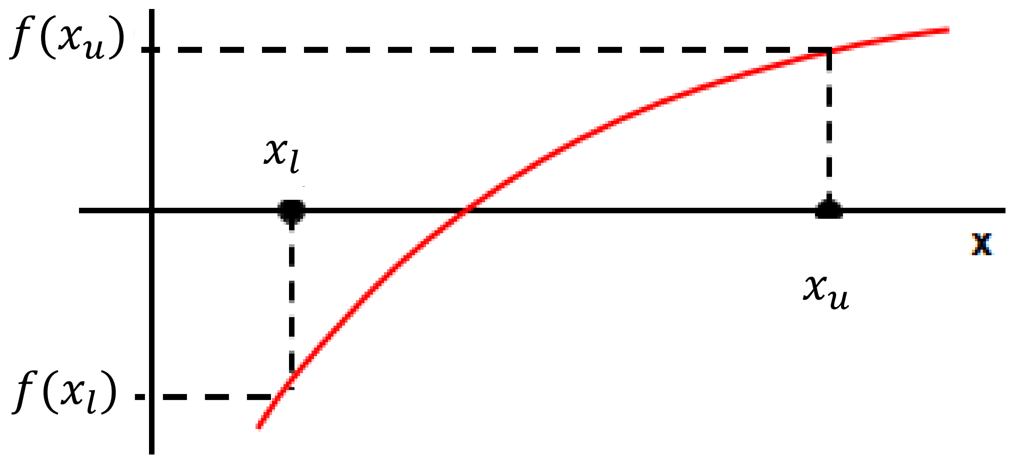


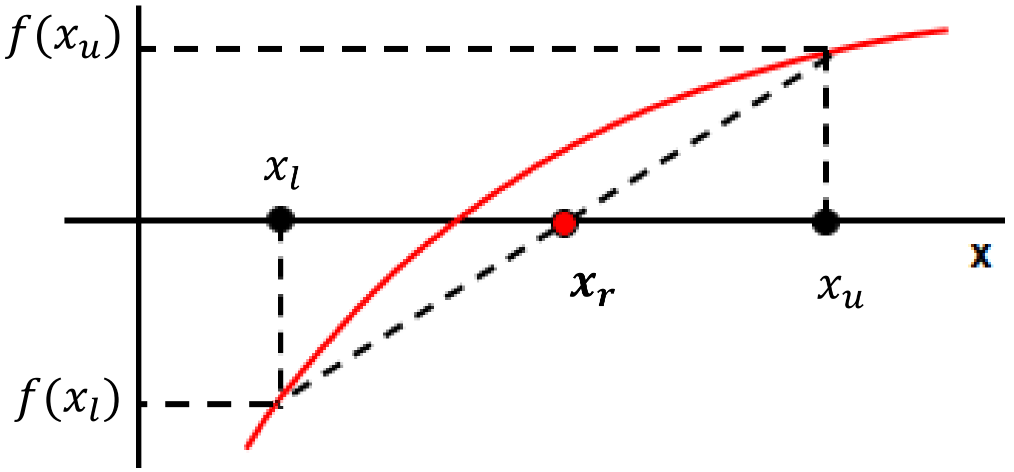
The fact that the replacement of the curve by a straight line gives a “false position” of the root is the origin of the name, *method of false position,* or in Latin, *regula falsi.* It is also called the *linear interpolation method.*

Using similar triangles

$$\boxed{x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}}$$

FALSE-POSITION METHOD





- **Step 1:** Choose two initial estimations, x_{LOWER} (x_l) and x_{UPPER} (x_u).
- They should bracket the root, i.e.

$$f(x_l) * f(x_u) < 0$$

- **Step 2:** Using similar triangles, the intersection of the straight line with the x axis can be estimated as

$$x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)}$$

or

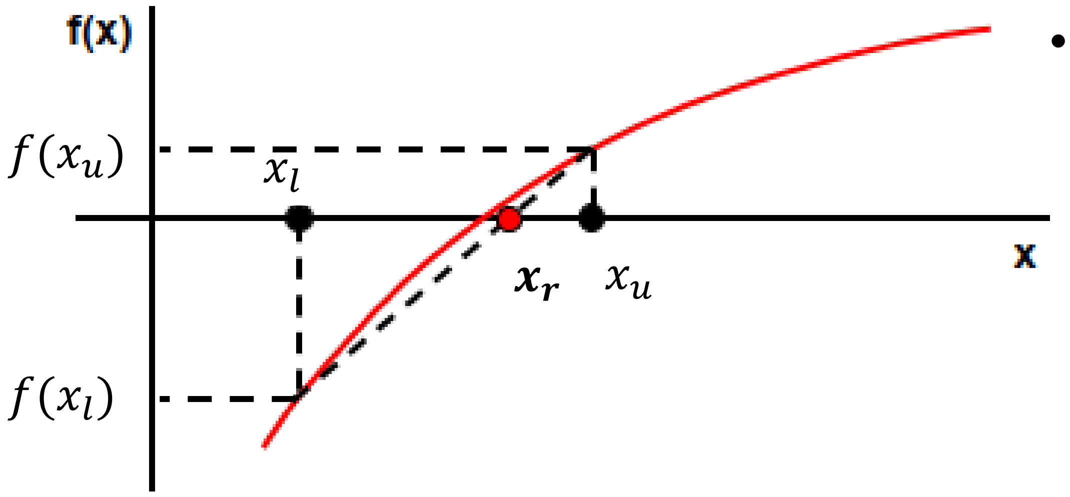
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

- Determine the interval which contains the root:

  IF $f(x_l) * f(x_r) < 0$ root is between x_l and x_r

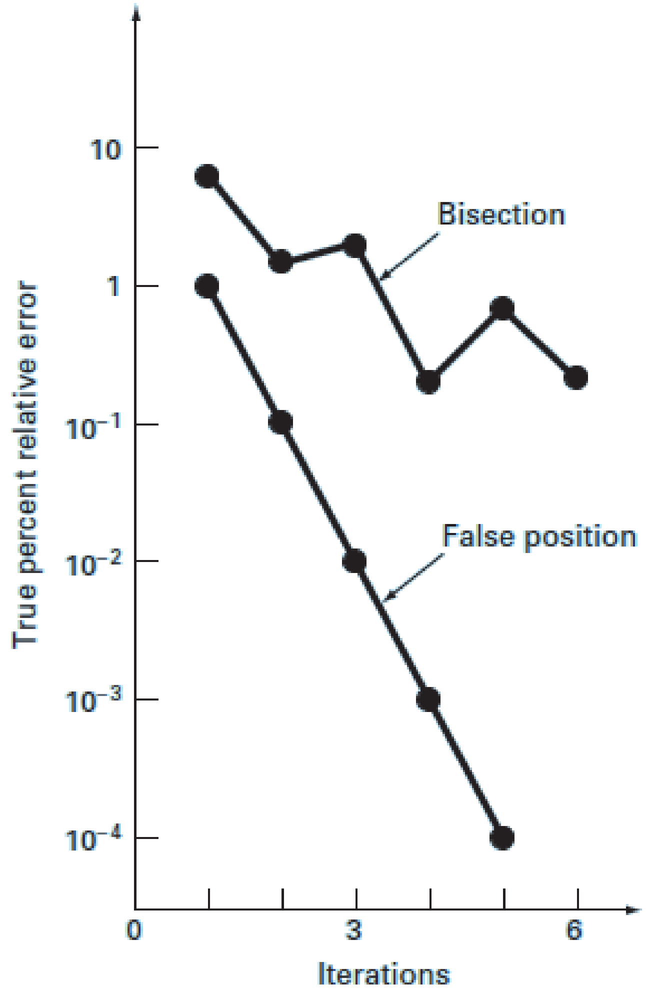
  ELSE root is between x_r and x_u

3.3.3. FALSE-POSITION METHOD



- **Step 3:** Estimate a new root in this interval,

  Stop when the specified tolerance is reached.



- False-position method always converges to the true root,
- $f(x_l) * f(x_u) < 0$ is true if the interval has odd number of roots, not necessarily one root.
- The *false-position method* generally converges faster than the *bisection method*.

Example 3.2: Find the square root of 11 by using false-position method. (Tolerance value: $|\varepsilon_s| = 0.5\%$)

$$x^2 = 11 \;\rightarrow f(x) = x^2 - 11 \quad (exact\ solution\ is\ 3.31662479)$$

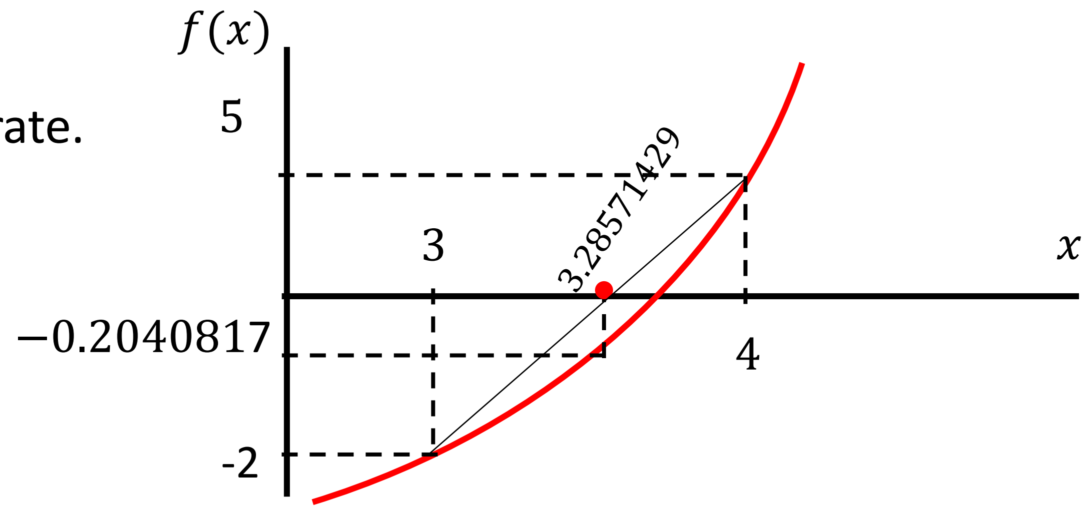
Choose initial estimates: $3^2 = 9 < 11$ and $4^2 = 16 > 11 \;\rightarrow \boldsymbol{x_l = 3}$ and $\boldsymbol{x_u = 4}$

1st iteration: $x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.28571429$ and

$$f(3.28571429) = 3.28571429^2 - 11 = -0.2040817$$

$f(x_l) * f(x_r) = -2 * -0.2040817 \rightarrow 0.408163 > 0$ (root is between x_l and x_r)

SET $x_l = x_r$= 3.28571429 and CONTINUE to iterate.



2nd iteration: $\boldsymbol{x_l = 3.28571429, x_u = 4}$

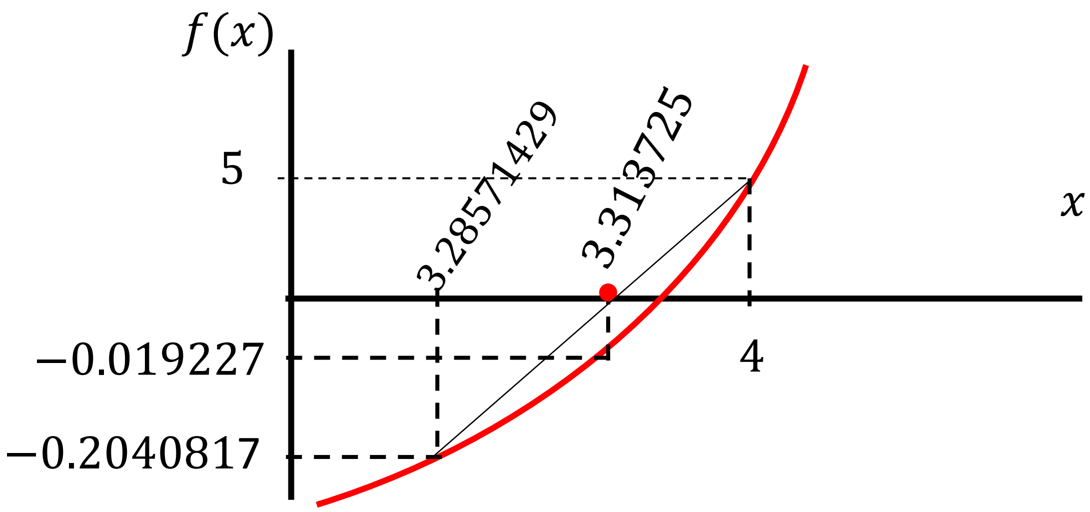
$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.313725$ and

$f(3.313725) = 3.313725^2 - 11 = -0.019227$

$f(x_l) * f(x_r) = -0.2040817 * -0.019227$

$\rightarrow 0.00392 > 0$ (root is between x_r and x_u)

SET $x_l = x_r$= 3.313725 and CONTINUE to iterate.



3rd iteration: $\boldsymbol{x_l = 3.313725, x_u = 4}$

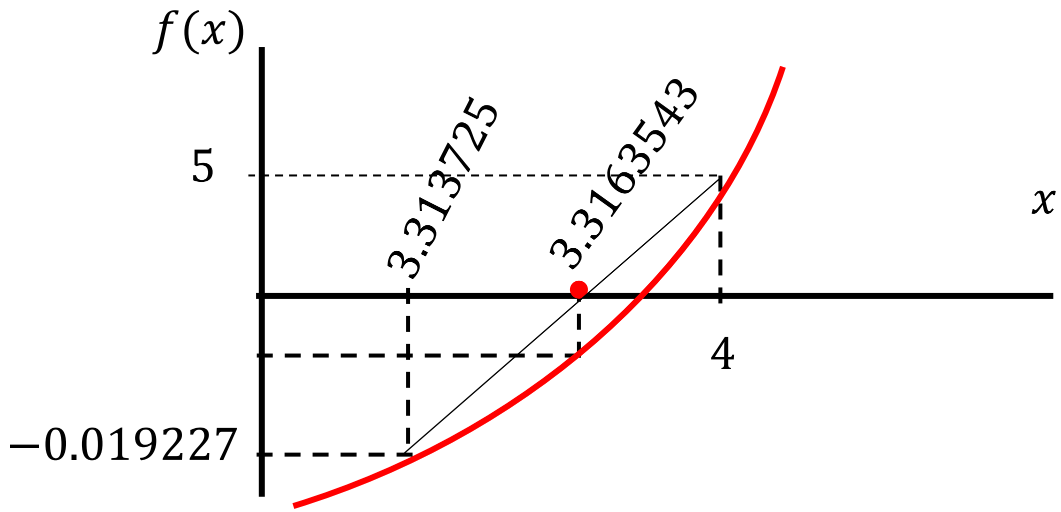
$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.3163543$ and

$f(3.3163543) = 3.3163543^2 - 11 = -0.00179415$

$f(x_l) * f(x_r) = -0.019227 * -0.00179415$

$\rightarrow 0.0000344961 > 0$ (root is between x_r and x_u)

SET $x_l = x_r$=3.3163543 and CONTINUE to iterate.



4th iteration: $x_l = 3.3163543, x_u = 4$

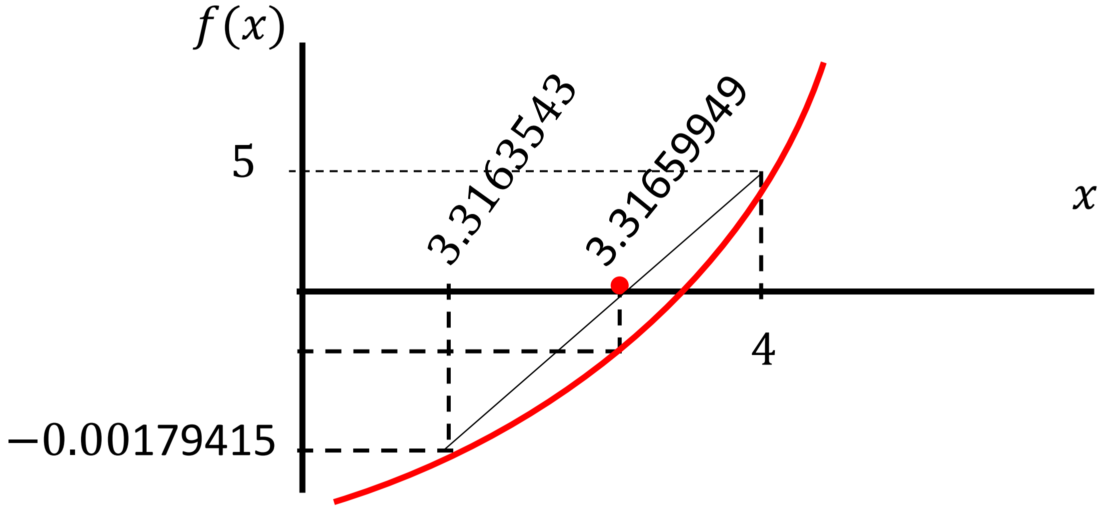
$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.31659949$ and

$f(3.31659949) = 3.31659949^2 - 11 = -0.0001678$

$f(x_l) * f(x_r) = -0.00179415 * -0.0001678$

$\rightarrow 0.0000344961 > 0$ (root is between x_r and x_u)

SET $x_l = x_r$=3.3163543 and CONTINUE to iterate.



Iteration	x_l	x_r	x_u	$f(x_l)$	$f(x_u)$	$f(x_r)$	$f(x_l)f(x_r)$	$\lvert\varepsilon_a\rvert$ (%)
1	3	3.285714	4	-2	5	-0.2040816	0.4081632653	--
2	3.285714	3.313725	4	-0.20408	5	-0.0192234	0.0039231379	0.84531
3	3.313725	$\boxed{3.316354}$	4	-0.01922	5	-0.0017969	0.0000345424	$\boxed{0.07926}$
4	3.316354	3.316599	4	-0.0018	5	-0.0001678	0.0000003016	0.00741
5	3.316599	3.316622	4	-0.00017	5	-0.0000157	0.0000000026	0.00069
6	3.316622	3.316625	4	-1.6E-05	5	-0.0000015	0.0000000000	0.00006

- Note that false-position method converged faster than the bisection method.

Notes on Bracketing Methods

- A plot of the function is always helpful.
  - To determine the number of all roots, if there are any.
  - to determine whether the roots are multiple or not.
  - to determine whether to method converges to the desired root.
  - to determine the initial guesses.
- Incremental search technique can be used to determine the initial guesses.
  - Start from one end of the region of interest.
  - Evaluate the function at specified intervals.
  - If the sign of the function changes, than there is a root in that interval.
  - Select your intervals small, otherwise you may miss some of the roots. But if they are too small, incremental search might become too costly.
  - Incremental search, just by itself, can be used as a root finding technique with very small intervals (not efficient).

NEXT WEEK
ROOTS OF EQUATIONS
Open Methods