Yayın No=8

Tr. J. of Engineering and Environmental Sciences
21 (1997) 101-117

Comparison of Simulation Effectiveness of Motion Equation Generation Methods for Articulated Rigid Body Systems

Sedat BAYSEÇ , Sadettin KAPUCU
University of Gaziantep, Mechanical Engineering Department, Gaziantep-TURKEY

Received: 26 / 2 / 1996

Abstract: Mathematical models of robotic systems not only provide the designer with the most valuable resource on which ideas about control and motion design can be tested or the performance of a non existing sketch can be checked before a prototype is manufactured, but also provide a versatile teaching and practising aid. The mathematical model of the system includes the characteristics of the manipulator and its drive systems and is definitive of their dynamic behaviour. In most robotic applications, the response of the controller is much faster than the drives and the manipulator and hence its model is generally not required. This paper presents a treatise on the generation of motion equations of robotic manipulators, excluding the drives. Newton-Euler, Lagrange and Hamilton equations and a formulation based on gradient methods are examined and tested on an all-revolute, three degrees of freedom planar articulated linkage. The resulting closed form motion equations are presented for comparison, and the rules of thumb for the use of these equations are given. Finally, some of the well known general purpose simulator softwares, some of which are commercially available, are described.

Key Words: Robot manipulators, Newton-Euler equations, Lagrange's equation, Hamilton's Equations, Gradient methods.

Artiküle Rijit Cisim Sistemleri İçin Hareket Denklemleri Türetme Metodlarının Simülasyona Yönelik Karşılaştırılması

Özet: Robotik sistemlerin matematik modelleri tasarımcıların kontrol ve hareket profili belirlemede fikirlerini uzerinde uyguladıkları önemli bir araç olup aynı zamanda eğitim amaçları içinde kullanılırlar. Matematik model sistemin karakteristiklerini içerip dinamik davranışını belirleyicidir. Robotik uygulamaların çoğunluğunda kontrol donanım ve yazılımının işlem hızı maniplatör ve sürücü motorlarından çok daha yüksek olduğundan bunların modellemesi genellikle aranmaz. Bu makale robot maniplatörlerinin sürücü motorları dışında olmak üzere hareket denklemlerinin elde edilmesi konusunda genel bir bilgi sunmaktadır. Newton-Euler, Lagrange ve Hamilton formulleri ile gradient metodları ile ilgili bir başka formülasyon incelenmiş ve 3 serbestlik dereceli, döner eklemlerden oluşan düzlemsel bir maniplatörün modellemesinde kullanılmıştır. Türetilen hareket denklemleri karşılaştırma için sunularak sözü edilen metodlarla ilgili uygulama esasları anlatılmıştır. Son olarak bir kısmı ticari olarak mevcut genel amaçlı simülatör programları tanıtılmıştır.

Anahtar Sözcükler: Robot maniplatörler, Newton-Euler denklemleri, Lagrange denklemi, Hamilton denklemleri, Gradient metodları.

Introduction

Robotic manipulators are composed of rigid links in articulation, connected to one another by lower kinematic pairs. They differ from mechanisms with their extended movabilities and therefore require no functional closed link loops. To provide solid positioning in space a manipulator has to have 6 degrees of freedom, at least 3 of which have to be rotational. It has to be simple in construction and control. Because of higher contact or bearing stresses and the requirement of packing multiple servo motors and power transmission elements in a small volume, multi-degrees of freedom joints are not preferred. Universally accepted manipulator configuration is the connection of movable rigid links equal in number to the total degrees of freedom to one another by single degree of freedom, single input-single output joints, namely revolute and prismatic, forming an articulated open chain linkage. Each joint is actuated by a motor of its own, enabling all the succeeding links to move relative to the preceding link. Motors are controlled by a digital computer in the case of a robot, or by a human operator in the case of a tele-operator. Using a robot at the limits of

its capacity is a difficult task. On one hand, working conditions are variable with changing payloads and irregular kinematic requirements, and on the other hand, the dynamic behaviour and tendencies of a robotic system is hardly ever appreciable or predictable with so many coupled movabilities. Applications like repairwork at hazardous media need precise and reliable robot operation. A digital or analog simulation can be tested beforehand, which supposedly should display how the actual robot will behave on duty. Ideas on motion design or control can be implemented on a simulation initially to observe or at least get an idea about how the real robot will respond. Finally a fast animated simulation can be used as a teaching aid in practising to use a robot.

Simulations are based on mathematical models which incorporate the parameters of the real systems they simulate and are definitive of their dynamic behaviour. A simulation should respond to external forces in the same manner as the real system does. Compatibility of the simulation to the real system it simulates must be complete in kinematic, static and dynamic aspects. Mathematical models occur in the form of multiple non-linear differential equations, each of the order 2 or more and require their simultaneous solution. Closed form solutions of motion equations are never possible, therefore, a stage of numerical or analog integration is required. Numerical integration is slow, but more accurate than analog. However, real time integration of complicated equations are only possible only by analog means. Another important criterion in the selection of digital or analog means of implementation is the limitation in the number of arithmetic making modules of analog computers. In preparing a mathematical model computational efficiency is vital. Definition of efficiency may be dominated by accuracy, speed, number of individual arithmetic operations or integrations or a combination of these. At this point, one has to know different approaches to generate the motion equations of real systems and the advantages of each over the others. This paper aims to put forth a treatise on the generation of motion equations of articulated open chain linkages. As the operation speed of the control hard and software is much higher than that of the servo drives and the manipulator linkage, its model is hardly ever required. Simulation of servo drives whether electrical or hydraulic is a complete problem in itself.

Dynamics of Mechanical Systems

The dynamics of a system is the relationship between its kinematics and the forces acting on it. The first correct definition of the principles governing rigid body dynamics was published by Newton and comprises the basis of classical mechanics. His deductions known as "*Newton's laws*" are completely based on experimental observations and have no mathematical proof, but there has been no incident reported that does not follow these laws. Therefore they can be regarded as *axioms.*

Newton's Laws:

Written in three statements, they describe the dynamics of particle motion. The first indication that these laws could be used to define angular motion is due to Euler and hence the approach as a whole is known as the Newton-Euler formula.

First law:

Newton's first law states that every *material body* remains in its state of rest or uniform rectilinear motion unless a net non-zero force acts on it. In conjunction with the first law, Newton defines the *quantity of motion* as the product of two factors, the velocity and the *quantity of matter*, that is, the *linear momentum* as:

$$P = mv \tag{1}$$

where m is the mass of the body, v is its rectilinear velocity and P is the linear momentum. For a body having constant mass and velocity, momentum remains unchanged, hence the first law is generally known as *the law of the conservation of momentum.*

The first law can be applied to the angular motion of a rigid body as:

$$K = I\omega \tag{2}$$

where I is the mass moment of inertia of the body, ω is its angular velocity and K is the angular momentum. On a constant inertia body, if no external moments act, it conserves its angular velocity and hence angular momentum.

For static compatibility, a physical model and the accompanying mathematics must obey the first law. Application of the first law to a system of bodies require the setting up of some constraint equations, which describe the link dimensions and types of joints and the connectivities they provide. Static compatibility is based on kinematic compatibility described by the constraint equations. Equations 1 and 2 are vector equations, meaning that the expressions on both sides of the equation are equal in magnitude and direction.

Second Law:

Newton's second law describes the general motion of rigid bodies, stating that the change in the *quantity of motion* is equal to the net force acting on it and takes place in the direction of the straight line along which the force acts. *Change* is meant with respect to time and so:

$$\dot{P} = F \tag{3}$$

where F is the resultant vector of all external forces acting. The idea can be extended for rotating bodies so that the rate of change of angular momentum is equal to the net moment acting as:

$$\dot{K} = \tau \tag{4}$$

where τ is the resultant moment or torque.

The second law defines how the motion of a body proceeds in time under the effect of a given forcing system. For dynamic compatibility, a model must obey the second law.

Third Law:

Newton's third law states that action always equals reaction. The forces that two bodies exert on each other are equal in magnitude and opposite in direction. The third law, defining the nature of the force interaction between different bodies makes possible the transition from the mechanics of single bodies into compound systems. The third law concerns the kinematic and static compatibilities.

Newton's laws completely define the motion of bodies or systems of bodies as a function of time. Application of the laws for planar linkages leads to the systematic free-body definition of system subsets. Each free body moving in a plane has 3 degrees of freedom and 3 equations can be derived for each movability. Between the 3N coordinates defined for the system having N bodies and n degrees of freedom, (3N-n) many are related with each other leaving only n independent generalised coordinates. Existence of constraints creates two problems in application: holonomic constraints between the coordinates defined and time in some cases add that many algebraic equations to be solved with the differential equations of motion simultaneously. This difficulty can be overcome by deriving the motion equations using the generalised coordinates, resulting in fewer but more complicated equations. Secondly, the force and motion constraints at the joints appear in the form of non-holonomic constraints which can not be integrated alone, and thus, must be solved with the rest of the motion equations simultaneously. This process yields the forces interacting at the joints, which may or may not be required.

Starting with these fundamental laws, alternative techniques were developed to eliminate the necessity of obtaining explicit expressions for the constraint forces, such as Lagrange's and Hamilton's equations.

Lagrange's Equation

Lagrange's equation for holonomic systems:

Equations of motion for dynamic systems can be specified in the form of Lagrange's equation as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j \tag{5}$$

$$j=1, \ldots n$$

where L is the Lagrangian, q_j are the generalised coordinates and Q_j is the generalised force acting on the j'th generalised coordinate. By definition the Lagrangian of the system is:

$$L = T - V \tag{6}$$

where T is the total kinetic energy and V is the total potential energy of the system. Lagrange equation describes the dynamics of the associated coordinate only. Therefore for a system of n degrees of freedom, n equations are derived which simultaneously define the dynamics of the whole system.

There is complete freedom in the choice of generalised coordinates as long as they are independent of each other, so that the kinematics of the system are uniquely defined.

The Lagrangian of the system is the difference between the kinetic and potential energies, both of which are scalar quantities. Therefore the Lagrangian of a system will have the same value for a given condition as long as the same definition is used in each case, because the lagrangian of a system is not unique. If L (q, $\dot{q}$, t) is an appropriate Lagrangian and F (q, t) is any differentiable function of the generalised coordinates and time, then

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + dF/dt \tag{7}$$

is also a Lagrangian of the system. For example the gravitational potential energy is defined with respect

to a reference datum. If the datum is changed for the same system, both the analytical form and the numerical value of the Lagrangian changes.

The generalised force Q_j is the net effect of all the external forces on the j'th generalised coordinate, thus,

$$Q_j = \sum_k F_k \frac{\partial x_k}{\partial q_j} \tag{8}$$

where x are coordinates defining the position of the system in real and virtual displacements and F_k is the net external force applied on coordinate x_k. Generalised forces are composed of all the forces external to the system. These external forces can be arbitrary functions of the generalised coordinates and time. Physically, they may be forces involving an energy injection into the system such as actuator forces, or involving energy dissipation from the system such as the velocity dependent damping forces of viscous dampers and position,velocity and acceleration dependent forces due to Coulomb friction. They can further include forces exerted by potential fields not included in the Lagrangian such as weights and forces coming from energy storage devices like mechanical springs, air cylinders etc.

The concept of kinematic compatibility of mechanical networks is included in the Lagrangian and in the generalised forces. As Newton's equations, Lagrange's equations can be derived for any number of coordinates resulting in that many differential equations of motion. In conjunction with the constraint equations which are equal to the number of equations less the number of degrees of freedom of the system, the dynamics of the system is fully defined.

Lagrange's equation for non-holonomic systems:

Non-holonomic systems have constraints described by differential equations which can not be integrated independent of the system dynamics. Non-holonomic constraints apply constraint forces on the system to make it obey the constraints. With the inclusion of the constraint forces, Lagrange's equation becomes:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j + \sum_{i=1}^{(m-n)} \lambda_i \frac{\partial f_i}{\partial q_j} \tag{9}$$

$$j = 1,2,..m$$

where f_i are constraint equations, m is the number of coordinates and λ_i are *Lagrange's undetermined multipliers*. The constraint equations can be explicit functions of time, that is, rheonomous or may not be explicitly dependent on time, that is, scleronomous.

Hamilton's Equations

Equations of motion for dynamic systems can be specified in the form of the canonical equations of Hamilton as:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \tag{10}$$

$$\dot{p}_j - Q_j = -\frac{\partial H}{\partial q_j} \tag{11}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{12}$$

where Q_j are the generalised forces, H is the *Hamiltonian*, L is the *Lagrangian* and p_j are the *conjugate momenta*, functions of generalised coordinates q_j, generalised velocities $\dot{q}_j$ and time t as:

$$\frac{\partial L}{\partial \dot{q}_j} = p_j \tag{13}$$

Hamiltonian H is defined as:

$$H(q,p,t) = \sum_j \dot{q}_j p_j - L(q,\dot{q},t) \tag{14}$$

If Lagrangian is independent of time and the potential energy independent of velocities, the Hamiltonian becomes:

$$H = T + V \tag{15}$$

To apply the canonical equations of Hamilton to mechanical networks, first a set of generalised coordinates are defined and the Lagrangian is formulated. Then conjugate momenta are derived using equation 13. Once momenta and Lagrangian are known, the Hamiltonian is formulated and substituted into the canonical equations.

Hamilton's method produces 2n first order differential equations, n being the degrees of freedom. The first canonical equation is used to obtain velocities from momenta. The second equation incorporates the principles of dynamics. Hamiltonian formulations in the form of equations 10-12 can describe the conservative and holonomic systems. For non-conservative and non-

holonomic systems, the constraint equations must also be included in the set of equations of motion.

Approximate Dynamics

The method of dynamic analysis developed by Rooney and Rai, presented in their paper published in 1976 brings a different approach to the analytical definition of system dynamics. This work is based on the method developed by Jones on the simultaneous solution of multiple non-linear algebraic equations which was reported in his paper published in 1973. This approach is based on the generation of a set of solutions which obey Newton's second law,

$$\ddot{q}_j = \frac{1}{m_j} F_j \tag{16}$$

where F_j is the net force active in direction of the coordinate q_j and m_j are the relevant mass or moment of inertia. The number, m, of coordinates q_j used in defining the system dynamics may exceed the number of degrees of freedom n that the system possesses. This implies that a set of (m-n) kinematic constraints f exist between the coordinates, described in the form:

$$f_i(q_i, q_j, q_m) = 0 \qquad i = 1,2,.. (m-n) \tag{17}$$

The solution is also forced to follow the constraint equations. Various methods of this type, known as gradient methods exist. The one presented here drives the system accelerations in a direction as to satisfy equation 17 with a force proportional in magnitude and opposite in direction to the errors in the constraint conditions. Constraint function f_i are zero in exact equations, but are normally non-zero, but of small magnitude in approximate equations. The force component F_j in equation 16 therefore is composed of two terms, one the generalised force active on a specific coordinate due to externally applied actuation and dissipative forces Q_j and the other a constraint force P_j which is producing a corrective action in a direction so as to modify the relevant acceleration so that equation 17 is satisfied. Equation 16 can be rewritten as:

$$\ddot{q}_j = \frac{1}{m_j}(Q_j + P_j) \tag{18}$$

Q_j are of the same definition as in equation 8. P_j are the resultant of all constraint force components in direction q_j, that is:

$$P_j = \sum P_j \qquad i = 1,2,.. (m-n) \tag{19}$$

where P_{ji} is the i'th component of P_j acting on the i'th kinematic constraint defined by equation 17. It is proportional to the instantaneous magnitude of the relevant error function and in the opposite direction to it as:

$$p_j = -\beta \frac{\partial f_i}{\partial q_j} f_i \tag{20}$$

The proportionality constant β is a high number. In the limiting case, when β thends to infinity the error in f_i will tend to zero and the product βf_i will become equal to the undetermined multipliers in the Lagrange's equation in form of equation 9. The final form of equation 16 becomes:

$$\ddot{q}_j = \frac{1}{m_j}(Q_j - \beta \sum_i \frac{\partial f_i}{\partial q_j} f_i) \tag{21}$$

is equivalent to equation 9. Equation 21 in matrix form is:

$$\ddot{q} = m^{-1}[Q - \beta J^T f] \tag{22}$$

where $\ddot{q}$ is the acceleration vector, J^T is the transpose of the jacobian matrix of f_i, m^{-1} is the inverse of the mass or inertia matrix, Q is the vector of generalised forces acting or dissipative, and f is the vector of component errors in f_i space.

This method is easy to apply and equations generated are simpler than the exact equations in any form. There is a steady state error on constraints, that is equation 17 is never zero, but the appropriate selection of β reduces the errors to magnitudes practically negligible.

Comparison of Methods

Dynamics of a realistic system of inert bodies having n degrees of freedom can be formulated by n second order differential equations. The state of such a system can be described by any number of coordinates greater than or equal to the total degrees of freedom. If the number of coordinates defined, m, is greater than the degrees of freedom n, a set of (m-n) algebraic or differential constraint equations exist, defining the relation between the dependent coordinates. m is

arbitrary. For example, in the free body approach, for a system having N moving bodies, each moving body is assumed to have 3 degrees of freedom in plane, leading to 3N second order differential equations and (3N-n) constraint equations. Constraint equations can be algebraic or differential. Solution of differential and algebraic equations simultaneously develops great difficulty in self-formulating simulator programs. J.S.Rai indicated this problem in his Ph.D. thesis dated 1977, aimed at creating a general purpose program to simulate the dynamics of planar linkages. His solution approach was further developed by L.Chonggao in his Ph.D. work dated 1981 on the dynamic analysis of planar linkages, where he presented a software called *DIFALG* which can solve and integrate a mixed set of differential and algebraic equations.It is also possible to differentiate the algebraic constraint equations once or twice with respect to time to convert them into a form similar to the motion equations to provide uniformity in the format of the resulting equations, but automatic differentiation by computer is also a difficult task.

The number of unknowns and hence the arithmetic required in the solution of an algebraic equation can be reduced by the method of approximate dynamics developed by J.R.Jones. In his paper published in 1973, a dynamic solution for the kinematic constraint equations is assumed and the solution point is given as a high velocity proportional to and opposite in direction to its instantaneous error in reference to the constraints. Jones and Backhouse used the same approach in the solution of motion equations for a robot manipulator in their paper published in 1981.

Among the four methods briefly described, Newton's laws are direct and easy to apply. Generally applied to the free body description of the system, the numerical values for each variable during the computation can be easily interpretted and related to the instantaneous state of the system modelled. In addition to the difficulties of handling the constraint equations, application of Newton's laws on free bodies generate too many equations to solve. Forward dynamics, that is the calculation of the forces to provide a given set of kinematic conditions for a linkage of N links and J joints involves the simultaneous solution of (3N+2J) algebraic equations. Inverse dynamics requires the numerical integration of 3N second order differential equations, inverting a 3Nx3N matrix at each iteration to calculate the kinematics as a response to a given forcing system. 3N second order differential equations are broken into 6N first order equations in numerical integration. Though equations of motion are easy to derive, their solution is difficult and costly on a digital computer. On the other hand, the solution generates joint reaction forces which may or may not be a requirement. Reduction of the total number of differential equations of motion by eliminating some of the 3N coordinates through the use of generalised coordinates can be achieved by the application of Hamilton's or Lagrange's equations.

In Hamilton's approach the formulation of the equations of motion requires the differentiation of the Hamiltonian function with respect to the generalised momenta and generalised coordinates. The resulting equations are more complex than the ones obtained by Newton's laws, but since it produces fewer equations, numerical integration becomes easier. Integration of Hamilton's canonical equations give generalised displacements and momenta. Generalised momenta are of no use in simulation or control problems and are not as meaningful as the generalised coordinates or velocities. To extract velocities from momenta requires additional algebraic manipulations. In setting up the Hamiltonian, first the Lagrangian has to be formulated, therefore at this stage using Lagrange's method may be more convenient.

In Lagrange's formulation without multipliers, the number of motion equations is reduced to a minimum, which is equal to the number of free generalised coordinates or degrees of freedom. The derivation involves the differentiation of the derivatives of the Lagrangian with respect to generalised velocities with respect to time. Equations become large but their number becoming minimum is favorable in numerical integration. Forward dynamics requires the simultaneous solution of n algebraic equations for an n degrees of freedom system. Inverse dynamics needs the inversion of an nxn mass matrix and the numerical integration of 2n first order differential equations simultaneously. Lagrange's formulation with multipliers generates m second order differential equations together with (m-n) constraint equations for an n degree of freedom system. m can be anything greater than n. In this method, derivation of the equations becomes easier as more coordinates are defined but this increases the difficulty of solving them. Forward dynamics involves the solution of (2m-n) algebraic equations. Inverse dynamics requires the inversion of a (2m-n)x(2m-n) matrix and the integration of (3m-n) first order differential equations simultaneously. Lagrange multipliers are by-products of the solution and are measures of the constraint forces.

Approximate formulation of system dynamics differs from the exact formulations mentioned above. It can be utilised whenever least mathematics is required, to obtain faster solutions.

Formulation considerations

Unlike the closed loop structure of mechanisms, robot manipulators are made up of open chains and the formulation of their motion equations differs from that of mechanisms. The dynamic principles lying underneath are the same and motion equations can be derived using equations 1-22. Derivation of motion equations of open chain articulated linkages are simpler than that of mechanisms as their degrees of freedom in respect to the number of moving links is greater than mechanisms. For example, a 4 link all-revolute articulation contains 3 moving links and has 3 degrees of freedom. If the tip of the uppmost link is connected to the ground by a fourth revolute joint, the system becomes a 4-bar mechanism which has again 3 moving links but only 1 degree of movability. Rule of thumb is that, for the same system, if the number of differential equations increases, equations become simpler. In this respect, one should tend towards using Newton-Euler formulation. The easiest way of incorporating the system constraints is to differentiate the kinematic constraint equations with respect to time once or twice and put them into the same format as the motion equations. The most straightforward technique on the other hand is to use the Lagrange equation without multipliers.

Equations of motion can be generated to solve a forward or an inverse dynamics problem. In the inverse dynamics problem, the kinematic state of the system, that is, linear and angular positions, velocities and accelerations of each link are given and the requirement is to calculate the forces of actuation which will keep this given kinematic state. As all the derivative terms are given, differential motion equations become algebraic, where simultaneous solution gives the actuation forces. Solution of a forward dynamics problem involves the solution of algebraic equations and is therefore easy. In the forward dynamics problem, arbitrarily definable magnitudes of actuation forces are given and the resulting kinematics are sought. In this problem the linear and angular positions, velocities and accelerations of the links are all unknown. Initial values of positions and velocities should be given and hence the initial values of the accelerations are calculable. Motion must be developed in time in such a way that velocities are always equal to the time rates of change of positions and accelerations equal to the time rates of change of velocities. The solution of an inverse dynamics problem first requires the inversion of a mass or inertia matrix and then a numerical integration and therefore is more difficult. The same set of equations are used for both types of problems.

The derivation of motion equations are generally laborious and the possibility of making mistakes in the mathematics and computation is great. Once the computer implementation whether digital or analog is done, results should be checked for correctness and accuracy. Checking should never be done by using the same arithmetic approach used in the derivation of the motion equations, to prevent any possible repetition of mistakes. A kineto-static solution based on the D'alambert's principle with graphical solution of position, velocity, acceleration and force equations is probably the best means of checking. This method is easy to understand and apply, and as in the well illustrated books by J.Shigley, first published in 1961 and by R.Norton, published in 1992 are textbook materials now. A scaled stick-diagram of the mechanism comprises the position analysis. Velocity and acceleration polygons enable to grasp and get an insight into how the coordinated motion of each link is developing at that instant. As mathematics involved is minimum, the possibility of making human mistakes is negligible. Sample solutions must be carried out as many times as possible, enough to proove that the outputs of the computer simulation are correct.

Another way of checking for correctness is giving the conditions which will produce a known and expected motion profile. Examples to this are numerous. Motion along a vertical slideway for example is a free fall. Free motion of a link about a revolute joint is harmonic, with the natural frequency of the link. To hinder a dynamic link from moving in a multi degree of freedom system, it can be brought to its minimum potential energy state and assigned a very large mass which is initially at rest. This converts that particular link to a virtual ground. A conservative system keeps the level of its total energy, that is its Hamiltonian, constant. Kinetic and potential energy of individual links come up with complicated profiles, but the total kinetic and potential energies will vary in equal amounts but of opposite polarity, such that the total energy is conserved. If there are not any prismatic movements to infinite displacements, variation in total kinetic and potential energies will be periodic at a frequency equal to the system fundamental frequency. Energy injection or dissipation complicates the problem

and therefore should be avoided in the first stages of the tests for correctness.

Formulation Examples

After a brief description of some well known formulations used in the derivation of motion equations in the previous sections, formulation examples for a three degrees of freedom all-revolute system are given here to display the flow of derivations and also facilitate a comparison between the resulting equations, which all define the same linkage. The example system is a triple pendulum composed of 3 bobs with masses m1, m2, m3 and centroidal inertias I1, I2, I3 located at the tip of all revolute binary links of lengths l1, l2, l3 enumerated increasingly as outward from the fixed link. The angular positions of the links are measured as positive counterclockwise from the positive x axis of a right handed cartesian frame, origin at the fixed pivot and y axis in opposite direction to the gravitational acceleration g. Example motion is free of external forces or torques of actuation, hence system is conservative. The system is initially at rest at a position where all links are horizontal. System parameters are given in Figure 1. Motion develops as shown in this Figure. The profile of the angular positions in time is shown in Figure 2 for the first 5 seconds of the motion.

Formulation by Newton-Euler Equation

The triple pendulum under consideration and the free body diagrams of its 3 moving links are shown in Figure 3. Motion of each of these free bodies can be defined by applying equations 3 and 4 to each link. Masses and mass moments of inertia of the links are constant, hence the rate of change of linear momentum becomes equal to the product of mass and the

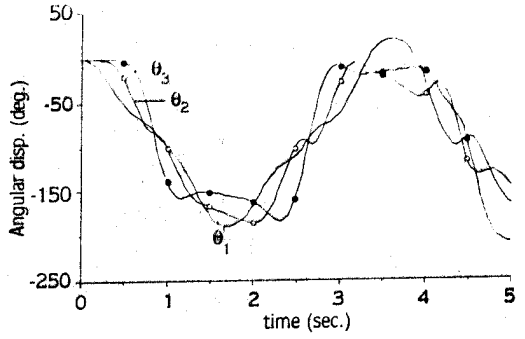


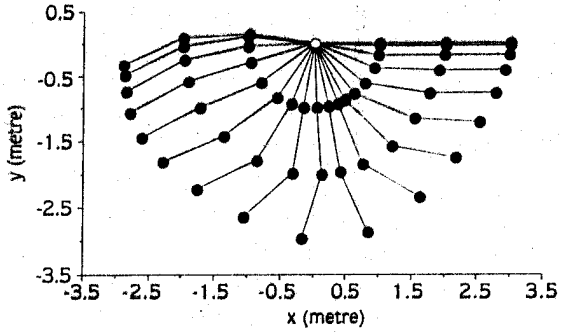
Figure 2. Profile of angular positions of the links of system in figure 1 for the first 5 seconds of its motion.

linear acceleration of the mass center. Linear accelerations of mass centers and the active forces can be separated into two components in x and y directions. Similarly the rate of change of angular momentum in equation 4 becomes equal to the product of the centroidal mass moment of inertia and the angular acceleration of the link. With the forces shown in Figure 3, the application of equations 3 and 4 yields motion equations for link 3 as:

$$m_3\ddot{x}_3 = F_{cx} \tag{23}$$

$$m_3\ddot{y}_3 = F_{cy} - m_3 g \tag{24}$$

$$I_3\ddot{\theta}_3 = \tau_3 + F_{cx}l_3 \sin\theta_3 - F_{cy}l_3 \cos\theta_3 \tag{25}$$



LINKAGE PARAMETERS:

L1=1.0 m.	m1=1.0 kg.	I1=0.0 kg-m^2
L2=1.0 m.	m2=1.0 kg.	I2=0.0 kg-m^2
L3=1.0 m.	m3=1.0 kg.	I3=0.0 kg-m^2

INITIAL CONDITIONS:	GEN. TORQUES:
THETA 1=0.0 deg.	T1=0.0 N-m.
THETA 2=0.0 deg.	T2=0.0 N-m.
THETA 3=0.0 deg.	T3=0.0 N-m.

THETA 1 DOT=0.0 rad/sec.
THETA 2 DOT=0.0 rad/sec.
THETA 3 DOT=0.0 rad/sec.

Figure 1. Chrono-cyclograph of a triple pendulum for the first 1.7 seconds of its motion, in 0.1 second intervals. System is released from rest from an initial position where all the links are horizontal. System parameters and initial values of generalised coordinates and coordinate velocities are shown aside.

and for link 2 as:

$$m_2\ddot{x}_2 = F_{bx}-F_{cx} \quad (26)$$

$$m_2\ddot{y}_2 = F_{by} - F_{cy}-m_2g \quad (27)$$

$$I_2\ddot{\theta}_2 = \tau_2-\tau_3 + F_{bx}l_2\sin\theta_2-F_{by}l_2\cos\theta_2 \quad (28)$$

and for link 1 as:

$$m_1\ddot{x}_1 = F_{ax}-F_{bx} \quad (29)$$

$$m_1\ddot{y}_1 = F_{ay} - F_{by}-m_1g \quad (30)$$

$$I_1\ddot{\theta}_1 = \tau_1-\tau_2+F_{ax}l_1\sin\theta_1-F_{ay}l_1\cos\theta_1 \quad (31)$$

The relationship between the locations of mass centers can be defined by using the following constraint equations:

$$x_3 = x_2 + l_3\cos\theta_3 \quad (32)$$

$$y_3 = y_2 + l_3\sin\theta_3 \quad (33)$$

$$x_2 = x_1 + l_2\cos\theta_2 \quad (34)$$

$$y_2 = y_1 + l_2\sin\theta_2 \quad (35)$$

$$x_1 = l_1\cos\theta_1 \quad (36)$$

$$y_1 = l_1\sin\theta_1 \quad (37)$$

Equations 23-37 together describe the motion of the whole system. These 15 equations must be simultaneously solved for accelerations and accelerations must be integrated to get velocities and positions. To convert the algebraic format of the constraint equations into a differential form like that of motion equations, they are differentiated with respect to time twice to take the following form:

$$\ddot{x}_2-\ddot{x}_3- \ddot{\theta}_3l_3\sin\theta_3=l_3\dot{\theta}_3^2\cos\theta_3 \quad (38)$$

$$\ddot{y}_2-\ddot{y}_3- \ddot{\theta}_3l_3\cos\theta_3=l_3\dot{\theta}_3^2\sin\theta_3 \quad (39)$$

$$\ddot{x}_1-\ddot{x}_2- \ddot{\theta}_2l_2\sin\theta_2=l_2\dot{\theta}_2^2\cos\theta_2 \quad (40)$$

$$\ddot{y}_1-\ddot{y}_2+\ddot{\theta}_2l_2\cos\theta_2=l_2\dot{\theta}_2^2\sin\theta_2 \quad (41)$$

$$\ddot{x}_1+\ddot{\theta}_1l_1\sin\theta_1=-l_1\dot{\theta}_1^2\cos\theta_1 \quad (42)$$

$$\ddot{y}_1-\ddot{\theta}_1l_1\cos\theta_1=-l_1\dot{\theta}_1^2\sin\theta_1 \quad (43)$$

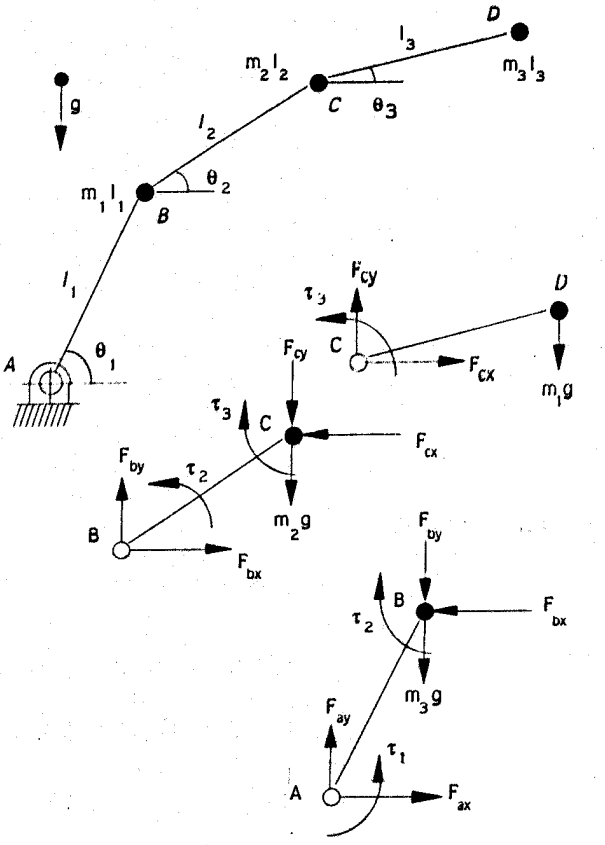


Figure 3. Triple pendulum and its links in free-body representation.

Substitution of the components of the joint forces defined by equations 23,24,26,27,29 and 30 into equations 25,28 and 31 eliminates them from equations describing link rotations to yield:

$$\ddot{x}_3(-m_3l_3\sin\theta_3)+\ddot{y}_3(m_3l_3\cos\theta_3)+\ddot{\theta}_3(I_3)= \tau_3-m_3gl_3\cos\theta_3 \quad (44)$$

$$\ddot{x}_2(-m_2l_2\sin\theta_2)+\ddot{y}_2(m_2l_2\cos\theta_2)+\ddot{x}_3(-m_3l_2\sin\theta_2)+ \ddot{y}_3(m_3l_2\cos\theta_2)+\ddot{\theta}_2(I_2)=\tau_2-\tau_3-(m_2+m_3)gl_2\cos\theta_2 \quad (45)$$

$$\ddot{x}_1(-m_1l_1\text{Sin}\,\theta_1)+\ddot{y}_1(m_1l_1\text{Cos}\,\theta_1)+\ddot{x}_2(-m_2l_1\text{Sin}\,\theta_1)+$$
$$\ddot{y}_2(m_2l_1\text{Cos}\,\theta_1)+\ddot{x}_3(-m_3l_1\text{Sin}\,\theta_1)+\ddot{y}_3(m_3l_1\text{Cos}\,\theta_1)+ \quad (46)$$
$$\ddot{\theta}_1(I_1)=\tau_1-\tau_2-(m_1+m_2+m_3)gl_1\text{Cos}\,\theta_1$$

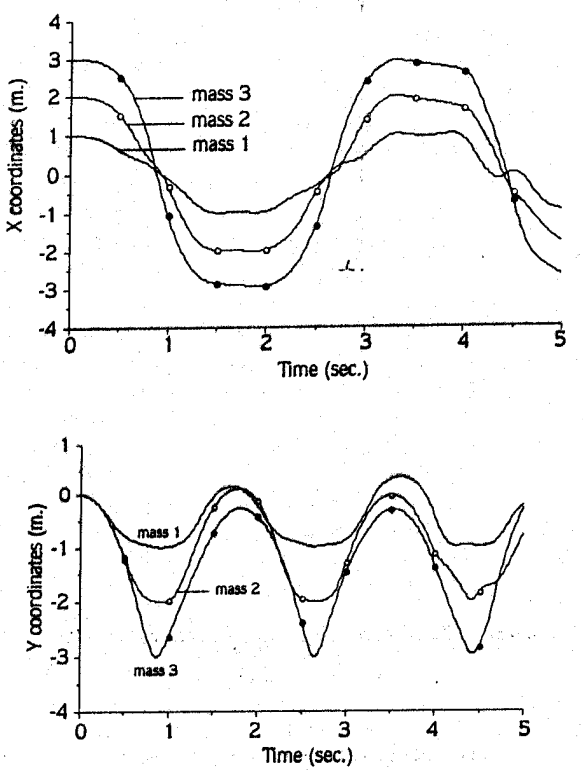


Figure 4. Profiles of x and y coordinates of the mass centers of the links of the triple pendulum shown in figure 1

Therefore, the resulting equations which describe the motion of the system uniquely, are equations 38-46. These equations can be solved for the accelerations in x_3, y_3, x_2, y_2, x_1, y_1, θ_3, θ_2 and θ_1, respectively, requiring the inversion of a 9x9 matrix. Numerical integration gives the profile shown in Figure 2 for link angles and the profiles shown in Figure 4 for the locations of the pendulum bobs. Linear accelerations described by equations 38-43 are definitive of joint force components. Figure 5 shows the x and y components of the linear accelerations of the mass centres produced by the integrating routine. Generation of joint forces is perhaps one advantage of the application of Newton-Euler equations. Another advantage is the simplicity in generating the motion equations.

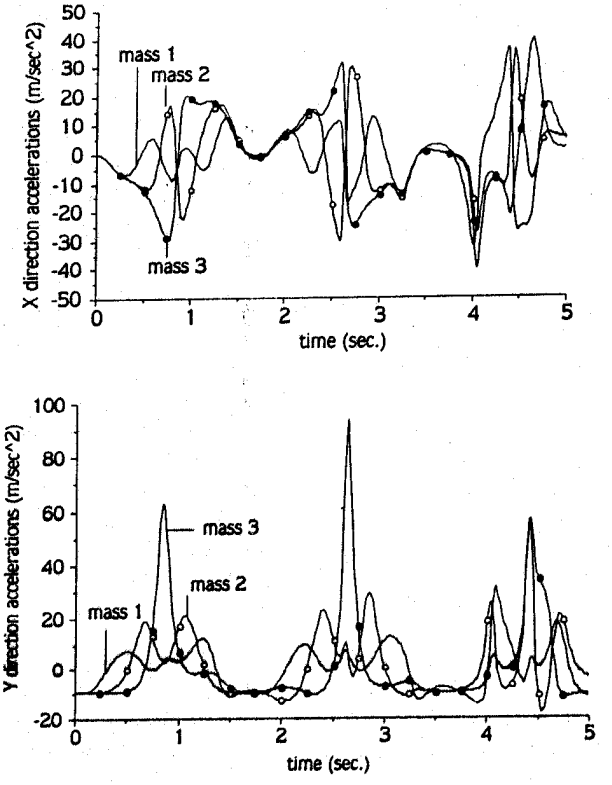


Figure 5. Profiles of x and y components of linear accelerations of the mass centers of the triple pendulum shown in figure 1. These acceleration components are definitive of the joint forces.

Formulation by Lagrange's Equation

The system under consideration is of 3 degrees of freedom hence derivation will end with 3 second order coupled equations each describing the motion of one degree of freedom or a generalised coordinate. Generalised coordinates are selected as θ_1,θ_2 and θ_3, the angles of links. Formulation starts with the derivation of the Lagrangian according to equation 6. All 3 moving links have masses and inertias, therefore the kinetic energy term T has 6 components as translational and rotational kinetic energy of each link summed up. V is the total potential energy term. Potential energy in a mechanical system can be due to elastic elements like balance springs or due to the gravitational potential field. These can either be included in the V of the Lagrangian or in the generalised forces Q of the Lagrange equation in the form specified by equation 8. In the system under consideration there are no elastic elements. 3 gravitational potential energy terms one

for each moving body are summed up to comprise the V term of the Lagrangian. Positions of m_1, m_2, m_3 are defined in cartesian coordinates as:

$$x_1 = l_1 \cos\theta_1 \qquad y_1 = l_1 \sin\theta_1 \tag{47}$$

$$x_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \qquad y_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \tag{48}$$

$$x_3 = l_1 \cos\theta_1 + l_2 \cos\theta_2 + l_3 \cos\theta_3 \qquad y_3 = l_1 \sin\theta_1 + l_2 \sin\theta_2 + l_3 \sin\theta_3 \tag{49}$$

The derivatives of equations 47-49 with respect to time give the cartesian components of the linear velocity of each mass. Velocities of mass centers therefore become:

$$v_1^2 = l_1^2 \dot{\theta}_1^2 \tag{50}$$

$$v_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \tag{51}$$

$$v_3^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + l_3^2 \dot{\theta}_3^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + 2 l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 \cos(\theta_1 - \theta_3) + 2 l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \cos(\theta_2 - \theta_3) \tag{52}$$

The y component of each mass is definitive of its potential energy with respect to the x axis passing through the fixed pivot. Lagrangian can be formulated as:

$$L = 0.5(m_1+m_2+m_3) l_1^2 \dot{\theta}_1^2 + 0.5(m_2+m_3) l_2^2 \dot{\theta}_2^2 + 0.5 m_3 l_3^2 \dot{\theta}_3^2 + (m_2+m_3) l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + m_3 l_1 l_3 \dot{\theta}_1 \dot{\theta}_3 \cos(\theta_1 - \theta_3) + m_3 l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \cos(\theta_2 - \theta_3) + 0.5 I_1 \dot{\theta}_1^2 + 0.5 I_2 \dot{\theta}_2^2 + 0.5 I_3 \dot{\theta}_3^2 - (m_1+m_2+m_3) g l_1 \sin\theta_1 - (m_2+m_3) g l_2 \sin\theta_2 - m_3 g l_3 \sin\theta_3 \tag{53}$$

Lagrangian given by equation 53 is processed through the differentiations of equation 5 to form the left hand side of that equation. Generalised forcing functions shown on the right hand side of equation 5 are torques τ_1, τ_2 and τ_3 directly acting on the generalised coordinates. As generalised coordinates are defined with respect to the ground, the reactions of the torques act on the ground. In reality, the reaction torque of a rotating actuator put to a joint acts on the preceding link. Therefore in this manner, torque on link 3 is τ_3 while net torques on moving links 1 and 2 become $\tau_1 - \tau_2$ and $\tau_2 - \tau_3$, respectively. The motion equations are put into a form where only the terms directly related to coordinate accelerations on one side of the equality sign and the remaining terms on the other side as:

$$[M] \cdot [\ddot{q}] = [Q + \phi] \tag{54}$$

where M is a symmetric, square matrix called the *mass or inertia matrix*. The concept of symmetry is given by:

$$M(i,j) = M(j,i) \tag{55}$$

This equation is a very important means of checking for correctness. $\ddot{q}$ is the ***acceleration vector*** and $Q+\phi$ is the *force vector* where Q is externally defined actuation and ϕ is the velocity dependent forces, namely *coriolis* and *centrifugal*. The equations of motion generated by Lagrange's formulation are therefore as follows:

For coordinate θ_1,

$$\ddot{\theta}_1[l_1^2(m_1+m_2+m_3)+I_1] + \ddot{\theta}_2[(m_2+m_3) l_1 l_2 \cos(\theta_1 - \theta_2)] + \ddot{\theta}_3[m_3 l_1 l_3 \cos(\theta_1 - \theta_3)] = -(m_2+m_3) l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - m_3 l_1 l_3 \dot{\theta}_3^2 \sin(\theta_1 - \theta_3) - (m_1+m_2+m_3) g l_1 \cos\theta_1 + \tau_1 - \tau_2 \tag{56}$$

For coordinate θ_2,

$$\ddot{\theta}_1[(m_2+m_3) l_1 l_2 \cos(\theta_1 - \theta_2)] + \ddot{\theta}_2[l_2^2(m_2+m_3)+I_2] + \ddot{\theta}_3[m_3 l_2 l_3 \cos(\theta_2 - \theta_3)] = (m_2+m_3) l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - m_3 l_2 l_3 \dot{\theta}_3^2 \sin(\theta_2 - \theta_3) - (m_2+m_3) g l_2 \cos\theta_2 + \tau_2 - \tau_3 \tag{57}$$

and for coordinate θ_3,

$$\ddot{\theta}_1[m_3 l_1 l_3 \cos(\theta_1-\theta_3)]+\ddot{\theta}_2[m_3 l_2 l_3 \cos(\theta_2-\theta_3)]+$$

$$\ddot{\theta}_3[l_3^2 m_3+I_3] = m_3 l_1 l_3 \dot{\theta}_1^2 \sin(\theta_1-\theta_3)+m_3 l_2 l_3 \dot{\theta}_2^2 \qquad (58)$$

$$\sin(\theta_2-\theta_3)-m_3 g l_3 \cos\theta_3+\tau_3$$

To solve these equations, both sides of equation 54 are pre-multiplied by the inverse of the mass matrix to yield the acceleration vector on the left and its numerical values on the right side of the equation. Numerical integration of equations 56,57 and 58 give the motion profile shown in Figure 2. Profiles for kinetic and potential energies and the total energy profiles are shown in Figure 6.

Formulation by Hamilton's Equation

As the system under consideration has 3 degrees of freedom, Hamilton's formulation will give 6 first order differential equations to define its dynamics. Generalised coordinates are θ_1,θ_2 and θ_3, the same as in Lagrange formulation. The first set of 3 equations describes the generalised or conjugate momenta p_1, p_2, p_3 relating to the coordinates θ_1,θ_2,θ_3, respectively, as functions of the generalised velocities. The second set of 3 equations describes system dynamics as defined by equation 11. The Lagrangian of the system as given by equation 53 is not an explicit function of time t and the potential energy terms incorporate only the terms due to gravity, and independent of velocities. The Hamiltonian of the system is equal to the sum of total kinetic and potential energies as indicated by equation 15.

$$H = 0.5(m_1+m_2+m_3)l_1^2\dot{\theta}_1^2+0.5(m_2+m_3)l_2^2\dot{\theta}_2^2+$$

$$0.5 m_3 l_3^2\dot{\theta}_3^2+(m_2+m_3)l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)+m_3 l_1 l_3 \qquad (59)$$

$$\dot{\theta}_1\dot{\theta}_3\cos(\theta_1-\theta_3)+m_3 l_2 l_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)+0.5 I_1\dot{\theta}_1^2$$

$$+0.5 I_2\dot{\theta}_2^2+0.5 I_3\dot{\theta}_3^2+(m_1+m_2+m_3)g l_1 \sin\theta_1+(m_2$$

$$+m_3)g l_2 \sin\theta_2+m_3 g l_3 \sin\theta_3$$

Variation of the Lagrangian of equation 53 with respect to the generalised velocities give generalised momenta as:

$$p_1=\dot{\theta}_1[I_1+(m_1+m_2+m_3)l_1^2]+\dot{\theta}_2[(m_2+m_3)l_1 l_2$$

$$\cos(\theta_1-\theta_2)]+\dot{\theta}_3[m_3 l_1 l_3\cos(\theta_1-\theta_3)] \qquad (60)$$

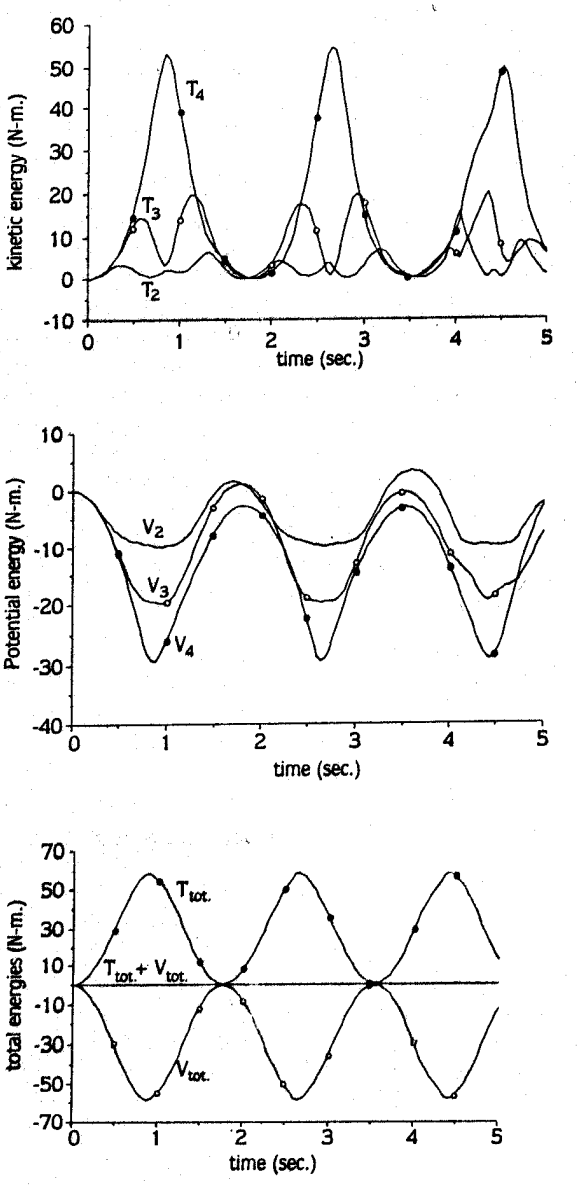


Figure 6. Kinetic and potential energy profiles of the links comprising the triple pendulum shown in figure 1. As system is conservative, total energy stays constant.

$$p_2=\dot{\theta}_1[(m_2+m_3)l_1 l_2\cos(\theta_1-\theta_2)]+$$

$$\dot{\theta}_2[I_2+(m_2+m_3)l_2^2]+\dot{\theta}_3[m_3 l_2 l_3\cos(\theta_2-\theta_3)] \qquad (61)$$

$$p_3=\dot{\theta}_1[m_3 l_1 l_3\cos(\theta_1-\theta_3)]+$$

$$\dot{\theta}_2[m_3 l_2 l_3\cos(\theta_2-\theta_3)]+\dot{\theta}_3[I_3+m_3 l_3^2] \qquad (62)$$

Variation of the Hamiltonian with respect to the generalised coordinates, when subtracted from the generalised forces, gives the time derivatives of the generalised momenta as:

$$\dot{p}_1=\tau_1-\tau_2-(m_2+m_3)\dot{\theta}_1\dot{\theta}_2 l_1 l_2 \sin(\theta_1-\theta_2)-m_3\dot{\theta}_1\dot{\theta}_3 l_1 l_3 \sin(\theta_1-\theta_3)-(m_1+m_2+m_3)g l_1 \cos\theta_1 \quad (63)$$

$$\dot{p}_2=\tau_2-\tau_3+(m_2+m_3)\dot{\theta}_1\dot{\theta}_2 l_1 l_2 \sin(\theta_1-\theta_2)-m_3\dot{\theta}_2\dot{\theta}_3 l_2 l_3 \sin(\theta_2-\theta_3)-(m_2+m_3)g l_2 \cos\theta_2 \quad (64)$$

$$\dot{p}_3=\tau_3+m_3\dot{\theta}_1\dot{\theta}_3 l_1 l_3 \sin(\theta_1-\theta_3)+m_3\dot{\theta}_2\dot{\theta}_3 l_2 l_3 \sin(\theta_2-\theta_3)-m_3 g l_3 \cos\theta_3 \quad (65)$$

Equations 60-65 are uniquely definitive of the system dynamics. In numerical integration, upon a given set of generalised momenta, equations 53-55 yield generalised velocities. Substitution of these calculated velocities and a given set of coordinates into equations 63-65 yields the time derivatives of the generalised momenta. Numerical integration produces the profile of the generalised coordinates as shown in Figure 2. The profile of the generalised velocities and momenta developes as shown in Figure 7.

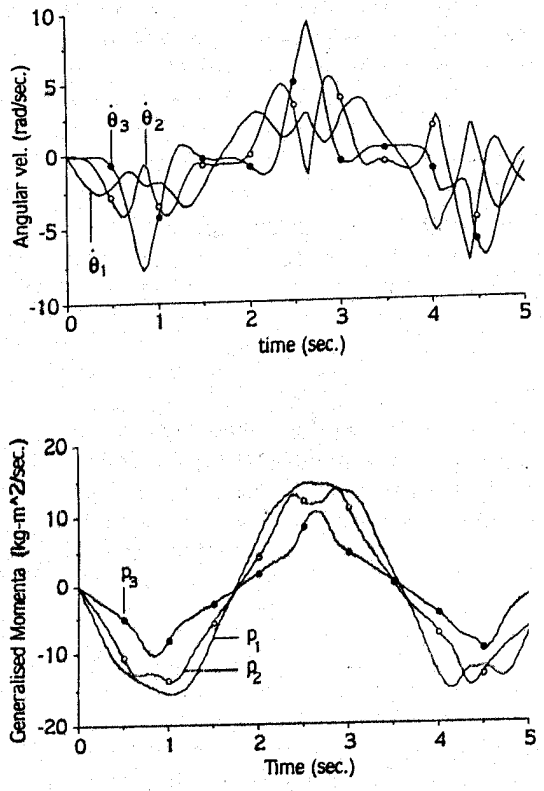


Figure 7. Profile of the generalised velocities and momenta for the system shown in figure 1.

Formulation of approximate dynamic equations

The kinematics of the triple pendulum under consideration can be represented by 7 coordinates as:

$$q = [\theta_1, \theta_2, \theta_3, x_2, y_2, x_3, y_3]^T \quad (66)$$

where θ are the absolute angles of the links as measured positively counterclockwise from the positive x axis, x_2 and y_2 are the coordinates of the mass center of link 2 and x_3 and y_3 are the coordinates of mass center of link 3. The system is of 3 degrees of freedom and hence there are 4 constraining equations:

$$\begin{aligned} f_1 &= x_2 - l_1\cos\theta_1 - l_2\cos\theta_2 \\ f_2 &= y_2 - l_1\sin\theta_1 - l_2\sin\theta_2 \\ f_3 &= x_3 - l_1\cos\theta_1 - l_2\cos\theta_2 - l_3\cos\theta_3 \\ f_4 &= y_3 - l_1\sin\theta_1 - l_2\sin\theta_2 - l_3\sin\theta_3 \end{aligned} \quad (67)$$

Magnitudes of the error functions in equation 67 are never zero in approximate dynamics, but can be reduced to magnitudes of fractions of a millimeter, which are negligible. The transpose of the jacobian matrix becomes:

$$J^T = \begin{vmatrix} l_1\sin\theta_1 & -l_1\cos\theta_1 & l_1\sin\theta_1 & -l_1\cos\theta_1 \\ l_2\sin\theta_2 & -l_2\cos\theta_2 & l_2\sin\theta_2 & -l_2\cos\theta_2 \\ 0 & 0 & l_3\sin\theta_3 & -l_3\cos\theta_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad (68)$$

The vector of generalised external forces is

$$Q = \begin{vmatrix} \tau_1-\tau_2-m_1 g l_1 \cos\theta_1 \\ \tau_2-\tau_3 \\ \tau_3 \\ 0 \\ -m_2 g \\ 0 \\ -m_3 g \end{vmatrix} \quad (69)$$

Substitution of equations 68 and 69 into equation 21 gives the following 7 second order equations to describe the system dynamics:

$$\ddot{\theta}_1=(\tau_1-\tau_2-m_1 g\cos\theta_1)/I_{A1}-\beta/I_{A1}\{l_1\sin\theta_1 (f_1+f_3)-l_1\cos\theta_1(f_2+f_4)\} \quad (70)$$

$$\ddot{\theta}_2=(\tau_2-\tau_3)/I_2-\beta/I_2\{l_2\sin\theta_2(f_1+f_3)- l_2\cos\theta_2(f_2+f_4)\} \quad (71)$$

$$\ddot{\theta}_3=\tau_3/I_3-\beta/I_3\{l_3 f_3\sin\theta_3-l_3 f_4\cos\theta_3\} \quad (72)$$

$$\ddot{x}_2 = -\beta f_1/m_2 \quad (73)$$

$$\ddot{y}_2 = -g -\beta f_2/m_2 \quad (74)$$

$$\ddot{x}_3 = -\beta f_3/m_3 \quad (75)$$

$$\ddot{y}_3 = -g -\beta f_4/m_3 \quad (76)$$

where I_{A1} is the mass moment of inertia of link 1 about the ground pivot. Equations 70-76 and the error functions of equation 67 form an equation set simpler than any form of exact equations. Integration of these equations are computationally not so difficult, but due to the existance of undetermined errors, it may take longer than necessary to integrate a set of exact equations. β should be assigned a large value. In the extreme, when β tends to infinity, the error functions of equation 67 tend to zero and the solution approaches to that of exact equations of motion. One difficulty likely to occur is that the solution under the control of infinitely large corrective constraint forces tends to go out of control, displaying high frequency and high amplitude oscillations on error functions. To prevent any such oscillations β must in practise be kept at moderate values, analogous to the *proportional gain* in a closed loop control system. This, however, increases the magnitude of errors which may go beyond tolerable limits. To keep β high and eliminate oscillations, a damping component can be included in the constraint force equations to transform equation 20 into:

$$P_j = -\beta\frac{\partial f_i}{\partial q_j}(f_i+\gamma\dot{f}_i) \quad (77)$$

where γ is an appropriate constant damping coefficient. This damping term does not affect the dynamics of the system, but only, dampens the constraint forces which is analogous to the *derivative gain* in a closed loop control system.

Profiles of coordinates obtained from the integration of equations 70-76 in the presence of the constraints of equation 67 are the same as that shown in Figures 2 and 4, without any visible or functional errors.

One important aspect in the selection of the technique to use is the time of computation required. As the number of equations increases, their numerical integration takes more time. Among the abovementioned techniques, the Newton formulation has produced 18 first order equations to integrate. Approximate dynamic formulation generated 14, Hamilton and Lagrange formulations generated 6 first order equations each. To integrate the equations defining the same physical system, with the same time duration and incrementation and with the same accuracy requirements, Hamilton and Lagrange formulations require almost the same amount of computation time. Approximate dynamic equations require about 1.3 times more and Newton formulation requires about 25 times more computation time. When the number of equations are tripled, one should expect a considerable amount of increase in computation time. The reason approximate dynamics requires such a short time is because it contains the least amount of mathematics. This makes it an efficient technique and with the least amount of mathematics involved, it becomes most suitable for analog computation or for use in preparing analog models for model referenced adaptive controllers.

Application Softwares

The first self-formulating program for the dynamics of mechanical systems was probably DYANA (Dynamic Analyser Programmer) produced by the General Motors Company. It was developed in 1958 to simulate one dimensional systems. Later, it was developed to include user-defined holonomic constraint equations.

From 1968, a general purpose program called DAMN (Dynamic Analysis of Mechanical Networks) was developed to simulate planar linkages by D.A. Smith and M.A. Chace at the University of Michigan. This program can handle dynamic or kinematic, constrained or unconstrained systems undergoing finite or infinitesimal displacements. Systems up to 30 links connected to each other by lower kinematic pairs can be modelled. Chonggao indicates a similarity between the way DAMN defines the linkage topology and Branin's technique for the automatic modelling of electrical networks in his 1981 study. Equations of motion are derived using Lagrange's formulation with multipliers. The number of equations developed is equal to the number of joints in the system. The constraint forces are calculated by an iterative determination of Lagrange multipliers. The program has a facility to accommodate user-defined forcing functions. Results of integration can be printed out or presented in form of graphs or simple mechanism stick diagrams drawn at a certain position.

DRAM (Dynamical Response of Articulated Machinery) is the second generation of DAMN. It uses D'alambert's principle to define the equations of motion. DAMN and DRAM are powerful dynamic programs which are commercially available. At the Central Electricity Generating Board of Britain, DRAM has been further modified and divided into 2 sections called AMP2D and AMP3D which can simulate planar and spatial mechanisms respectively. These programs can accomodate user-defined forcing functions and impact type forces, which are very difficult to integrate digitally. They both can solve forward and inverse dynamics problems.

Another commercially available program originating from the University of Michigan is ADAMS. Almost during the same period, between 1968 and 1971 another general purpose program to simulate mechanical networks named IMP (Integrated Mechanisms Program) was developed by J.J. Uicker Jr., D.F. Livermore and P.N. Sheth at the University of Wisconsin. IMP is based on the earlier work of Livermore, concisely described in his paper published in 1967. It was further developed by Sheth as a Ph.D. thesis and finally put into a commercially available multi purpose program as reported by Sheth and Uicker in their paper published in 1972. IMP can handle planar or spatial, multi-degrees of freedom, multi loop chains. The program uses the concepts of Graph Theory to define the mechanical network and formulates the constraint equations from the network topology. Constraint equations are used to generate a stiffness matrix which is substituted into the Hamilton's equations. The program can work in kinematic, static and dynamic modes, and hence can do static force analysis of structures as well. It can calculate joint forces using virtual work. Springs and dampers can easily be included into the simulation. Any other forcing functions can be defined externally. IMP has a powerful graphics package which can draw graphs or pictures of the network simulated and is commercially available.

Many other general purpose computer programs exist such as MEDUSA prepared by T.J. Lehman at the Illinois Institute of Technology, VECNET developed by G.C. Andrews and H.K. Kesavan at the University of Waterloo, SKINAL developed by Paul and Hud at the University of Pennsylvania, KIDYAN developed by Brat at the Czech Technical University, DAPL developed by G.T. Rooney and J.S. Rai at Liverpool Polytechnic and CADOM developed by H. Rankers at Delft University of Technology.

CATIA is an intricate and voluminous general purpose mechanisms simulation software prepared by a team of researchers in 1983, at Dassault Systems, France. It can handle both open and closed loop linkages. It has a graphics package which can present mechanisms in wire frame or detailed polyhedra representation and is graphically interactive. Basic robot tasks and related operations can also be implemented onto the simulation. While in motion, continuous checking of geometric incompatibilities and collisions are carried out and avoided. It can accomodate revolute and prismatic pairs. Any other joint types required are represented by a combination of these. Robot simulations allow up to 20 joints and robot systems up to 20 robots.

Another general purpose dynamic software prepared by Görür as an M.Sc. dissertation in 1989, initially aiming the simulation of internationally accepted comissioning tests of electric towers, can simulate the small and large scale displacements of structures due to external loads and material failure. In this software, the topology is described by the initial positions of the joints and stiffnesses between them. System equations

are automatically generated using Newton's laws. Moment equations for links are eliminated by converting the system into eqivalent masses concentrated at the joints. This layout enables the simulation of mechanisms and open chain linkages composed of all-revolute binary links also.

All of these programs are prepared to analyse mechanisms in general. Therefore, they are complex, slow and costly to use. Although many of them can analyse robotic manipulators, simpler programs were also made for only open-chains containing lower kinematic pairs. The versions used for control purposes like model referencing adaptive control aim towards real time solutions. A program prepared at the University of Florida by M.M. Thomas in 1981 defines a 6 degrees of freedom spatial open chain linkage with revolute joints only, with influence coefficients. Linkage dynamics are formulated by direct application of Lagrange's equation to do a forward dynamic analysis. Driving torques are assumed to be produced by rotary actuators placed at the joints. This program has limited applications as it can handle only all-revolute systems.

A robot simulator program developed by R. Featherstone of the University of Edinburgh does the inverse dynamics of n degrees of freedom articulated spatial open chains with revolute or prismatic pairs. As reported in 1982, the equations of motion are derived by Newton's laws applied for each link. The forcing functions at the joint axes are defined by the user in terms of coordinates, coordinate velocities and time. The program contains two control modules as coordinated and uncoordinated joint position control systems. It has a powerful graphics package for data output. B.K.P. Horn has described and formulated kinematics, statics and dynamics of a 2 degrees of freedom, all-revolute planar open chain in a well-explained paper published in 1979, where, kinematics is defined by vector chains. A static force analysis using Newton's first law is facilitated. Dynamic motion equations are derived by Lagrange's formulation.

The efficiency of digital simulation has been examined by Walker and Orin in their paper published in 1982 on the basis of the total number of mathematical operations required. The equations of motion for an n degrees of freedom spatial articulated open chain with prismatic and revolute joints are derived using Newton-Euler formulation. The constraint forces and moments occurring at joints are eliminated to simplify the solution. They also have presented techniques for solving joint accelerations.

Programs developed to simulate a variety of chain configurations contain constraint equations involving the types of movability of the individual joints. These equations are made to change and adapt to the condition to cater for the required constraints. They can be algebraic or differential, and can be solved either simultaneously with the differential equations of motion or kept in a correct state by iterative techniques.

Conclusion

Manipulator dynamics and control has become an important field of research. Formulation of motion equations is the first step in developing skills in control and sense of magnitude and understanding of the dynamic behaviour. Motion equations all stand on Newton's laws. Lagrange and Hamilton equations, which are derivable from Newton's laws, aim simpler to understand methods by dealing with energies. Gradient methods aim approximate solutions at which the solution is decending in the field of error functions along the steepest path, requiring the least amount of arithmetic for modelling, perhaps most suitable for analog modelling.

The theory underlying mathematical modelling is very old and related literature is extensive. All the theoretical work is now textbook material. What is existing less is application examples on open chain articulated linkages or more specifically, robot manipulators. Application of basic knowledge to open chains is not difficult, but needs know-how of its kind. This paper is aiming to be didactic, briefly presenting the basic knowledge covering 4 different kinds of formulations and application examples on the same mechanical system.

Triple pendulum is a planar system and has 3 degrees of freedom. When actuated by joint torques, it can be converted into a plane positioning manipulator. Normally a robot manipulator should be spatial with more degrees of freedom, but a triple pendulum was selected to display the application of the techniques as it is simpler and hence more instructive. Using the presented techniques and following the same procedure of formulation and with a bit more of labour, spatial systems can also be modelled.

No effort has been put into the simulation or mathematical modelling of drive systems as they are big problems in their own. Controlled torques calculated are normally substituted in place of the torque expressions in motion equations. This in turn completes the system into a more robot-like appearance.

References

Bradbury T.C., "Theoretical Mechanics", John Wiley and Sons, Inc. Library of Congress Catalog Card Number: 67-29932, 1968

Chace Milton A., "DAMN, a prototype program for the dynamic analysis of mechanical networks". University of Michigan publications, 1970, USA.

Chonggao, Liang, "Computer aided analysis of planar linkages". Ph.D. Dissertation, Liverpool Polytechnic, 1981, UK.

Featherstone, Roy, "Controlling of a robot arm", Department af artificial intelligence, DAI working paper no.68, University of Edinbourgh, 1981, UK.

Featherstone, Roy, "A program for simulating robot dynamics", Department af artificial intelligence, DAI working paper no.116, University of Edinbourgh, 1982, UK.

Goldstein Herbert, "Classical Mechanics". Addison Wesley Publishing Company, ISBN 0-201-02969-3,1980, USA.

Görür, Günseli, "Simulation of the dynamic response of planar, statically determinate trusses to external forces", M.Sc. dissertation, Middle East Technical University, Gaziantep, Turkey, 1989.

Horn, Berthold, K.P., "Kinematics, statics and dynamics of two dimensional manipulators", Artificial Intelligence, an MIT Perspective, Vol.2, pp.273-308, MIT Press, 1979.

"IMP Integrated Mechanisms Program", Structural Dynamics Research Corporation, Mechanical design library, User's manual. 1979, Ohio, USA.

Jones, John Rees, "Curve following in n-dimensional euclidean space". Simulation, Vol.21, No.5, pp.145-149, 1973.

Jones, J.R, Backhouse C., "Analog computer simulation of a robot manipulator". Journal of Mechanical Engineering Science, I. Mech. E. Vol.23, No.3, pp.121-129, 1981, UK.

Livermore, D.F., "The determination of equilibrium configurations of spring restrained mechanisms using (4x4) matrix methods", Journal of Engineering for Industry, Transactions of ASME, Ser.B, Vol.89, pp.112-118, 1967.

Norton, Robert L., "Design of Machinery", Mc Graw Hill Inc. ISBN: 0-07-909702-2, 1992.

Rai, J.S., "Dynamic analysis of planar linkages". Ph.D. Dissertation, Liverpool Polytechnic, 1977, UK.

Rooney G.T., Rai, J.S., "Approximate dynamic simulation of constrained mechanical systems", Simulation, Vol.27, No.6, pp.55-61, 1976.

Sheth P.N., "On a general purpose computer aided design analysis system for mechanisms", Ph.D. thesis, University of Wisconsin, Madison, 1971, USA.

Sheth, P.N., Uicker, J.J.Jr., "IMP (Integrated Mechanisms Program), a computer aided design analysis system for mechanisms and linkages", Journal of Engineering for Industry, Transactions of ASME, Ser.B, Vol.94, pp.454-464, 1972,

Shigley, Joseph Edward, "Dynamic Analysis of Machines", Mc Graw Hill Book Company Inc., ISBN: 07-056858-8, 1961.

Thomas, Mark M., "Dynamic model formulation for rigid link serial manipulators with revolute joints", Master of Engineering dissertation, University of florida, 1981.

Walker, M.W., Orin, D.E., "Efficient dynamic computer simulation of robot mechanisms", Journal of Dynamic Systems, Measurement and Control, Transactions of ASME, Vol.104, pp.205-211, 1982.

References

Bradbury T.C., "Theoretical Mechanics", John Wiley and Sons, Inc. Library of Congress Catalog Card Number: 67-29932, 1968

Chace Milton A., "DAMN, a prototype program for the dynamic analysis of mechanical networks". University of Michigan publications, 1970, USA.

Chonggao, Liang, "Computer aided analysis of planar linkages". Ph.D. Dissertation, Liverpool Polytechnic, 1981, UK.

Featherstone, Roy, "Controlling of a robot arm", Department af artificial intelligence, DAI working paper no.68, University of Edinbourgh, 1981, UK.

Featherstone, Roy, "A program for simulating robot dynamics", Department af artificial intelligence, DAI working paper no.116, University of Edinbourgh, 1982, UK.

Goldstein Herbert, "Classical Mechanics". Addison Wesley Publishing Company, ISBN 0-201-02969-3,1980, USA.

Görür, Günseli, "Simulation of the dynamic response of planar, statically determinate trusses to external forces", M.Sc. dissertation, Middle East Technical University, Gaziantep, Turkey, 1989.

Horn, Berthold, K.P., "Kinematics, statics and dynamics of two dimensional manipulators", Artificial Intelligence, an MIT Perspective, Vol.2, pp.273-308, MIT Press, 1979.

"IMP Integrated Mechanisms Program", Structural Dynamics Research Corporation, Mechanical design library, User's manual. 1979, Ohio, USA.

Jones, John Rees, "Curve following in n-dimensional euclidean space". Simulation, Vol.21, No.5, pp.145-149, 1973.

Jones, J.R, Backhouse C., "Analog computer simulation of a robot manipulator". Journal of Mechanical Engineering Science, I. Mech. E. Vol.23, No.3, pp.121-129, 1981, UK.

Livermore, D.F., "The determination of equilibrium configurations of spring restrained mechanisms using (4x4) matrix methods", Journal of Engineering for Industry, Transactions of ASME, Ser.B, Vol.89, pp.112-118, 1967.

Norton, Robert L., "Design of Machinery", Mc Graw Hill Inc. ISBN: 0-07-909702-2, 1992.

Rai, J.S., "Dynamic analysis of planar linkages". Ph.D. Dissertation, Liverpool Polytechnic, 1977, UK.

Rooney G.T., Rai, J.S., "Approximate dynamic simulation of constrained mechanical systems", Simulation, Vol.27, No.6, pp.55-61, 1976.

Sheth P.N., "On a general purpose computer aided design analysis system for mechanisms", Ph.D. thesis, University of Wisconsin, Madison, 1971, USA.

Sheth, P.N., Uicker, J.J.Jr., "IMP (Integrated Mechanisms Program), a computer aided design analysis system for mechanisms and linkages", Journal of Engineering for Industry, Transactions of ASME, Ser.B, Vol.94, pp.454-464, 1972,

Shigley, Joseph Edward, "Dynamic Analysis of Machines", Mc Graw Hill Book Company Inc., ISBN: 07-056858-8, 1961.

Thomas, Mark M., "Dynamic model formulation for rigid link serial manipulators with revolute joints", Master of Engineering dissertation, University of florida, 1981.

Walker, M.W., Orin, D.E., "Efficient dynamic computer simulation of robot mechanisms", Journal of Dynamic Systems, Measurement and Control, Transactions of ASME, Vol.104, pp.205-211, 1982.