

Chapter 4

Quantum Circuits

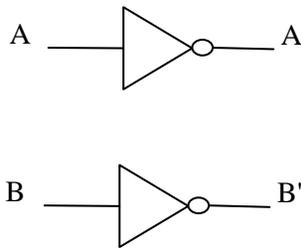
In the previous chapter we have investigated quantum gates. In this chapter we will construct and analyze quantum circuits.

A quantum circuit deals representation of operating principles of the multiqubit quantum computation.

In general a quantum gate represented by a unitary matrix. In usual a gate is operated by a single qubit input.

In the Boolean circuit the gates combined parallel or series in order to represent a digital circuit for a given problem. (i.e half-adder circuit). In order to understand difference between quantum and classical circuits let us consider parallel combination of gates. In a classical circuit if two one input gate (for example inverter) connected parallel then there are two bit input, if three inverter connected in parallel then there are three qubit input, so on.

Input		Output	
A	B	A'	B'
0	0	1	1
0	1	1	0
1	0	0	1
1	1	0	0



Truth Table. Linear combination of bits are not possible.

Part of a circuit. Two parallel inverter. The inputs are bits(1,0)

Truth table read as follows:
 A=0 then A'=1 B=0 then B'=1
 A=1 then A'=0 B=1 then B'=1

A quantum circuit diagram represent combination of quantum gates. Let us consider parallel combination of two single qubit gates (i.e combination of X gates). In contrast to classical circuit linear combinations of the qubits are possible. This property open a new way to the computational theory. As you see that parallel combination of two gate can be operated on 4 different input and output, parallel combination of 3 gate can be operated on 8 different input and output. So on. In general number of parallel gate, n, in a circuit and number of input and output of the circuit is related by:

$$\#input=2^n$$

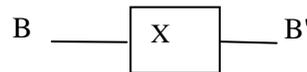
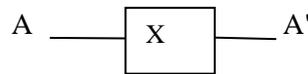
This relation is one of the most important property of the quantum computation.

In the example, input output relation can be obtained from the truth table,

- $|00\rangle \rightarrow |11\rangle$
- $|01\rangle \rightarrow |10\rangle$
- $|10\rangle \rightarrow |01\rangle$
- $|11\rangle \rightarrow |00\rangle$

As we will show later this can be expressed in terms of matrices.

Input		Output	
A	B	A'	B'
0	0	1	1
0	1	1	0
1	0	0	1
1	1	0	0



Truth Table. Linear combination of qubits are possible.

Part of a circuit. Two parallel quantum inverter. The inputs are qubits (0,1)

Important!!

Truth table read as follows:

The inputs A and B represent $|A\rangle|B\rangle = |AB\rangle$

The outputs A and B represent $|A'\rangle|B'\rangle = |A'B'\rangle$

Explicitly

$$|00\rangle \rightarrow |11\rangle$$

$$|01\rangle \rightarrow |10\rangle$$

$$|10\rangle \rightarrow |01\rangle$$

$$|11\rangle \rightarrow |00\rangle$$

Unitary matrix For a given quantum circuit

The quantum circuits can be represented by unitary matrices. Input of the circuits are qubits (quantum states) whose matrix representations are given by:

Single qubit:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Two qubit

$$|0\rangle \otimes |0\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad |0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix};$$

$$|1\rangle \otimes |0\rangle = |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad |1\rangle \otimes |1\rangle = |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

Three qubit

$$\begin{aligned}
|0\rangle \otimes |0\rangle \otimes |0\rangle = |000\rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; & |0\rangle \otimes |0\rangle \otimes |1\rangle = |001\rangle &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \\
|0\rangle \otimes |1\rangle \otimes |0\rangle = |010\rangle &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; & |0\rangle \otimes |1\rangle \otimes |1\rangle = |011\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \\
|1\rangle \otimes |0\rangle \otimes |0\rangle = |100\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; & |1\rangle \otimes |0\rangle \otimes |1\rangle = |101\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \\
|1\rangle \otimes |1\rangle \otimes |0\rangle = |110\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; & |1\rangle \otimes |1\rangle \otimes |1\rangle = |111\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};
\end{aligned}$$

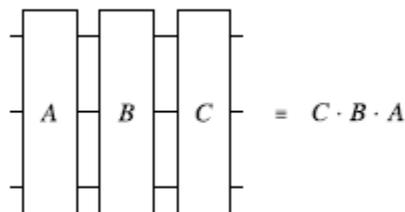
So on....

This matrix representation of the states are transformed by using transfer matrix of the gate. We point out again, single qubit represented by 1x2 matrix, two qubit represented by 1x4, three qubit represented 1x8 so on. This shows that one can present a circuit with n qubit that operates 2^n inputs and outputs.

Composing quantum gates in series

Quantum gates are represented by unitary matrices. When quantum gates connected series dot product of the matrix representations arises. Consider several gates say, A, B and C act upon the same subset of qubits then those gates must be applied in series and their overall action can be obtained from dot product of the matrices in reverse order:

$$C \cdot B \cdot A$$



The transfer matrix also can be obtained by constructing truth table. As an example consider the circuit. Pauli X- gate invert the input state. Hadamard gate present superposition of the states.

states. The truth table of the circuit read as follows:

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

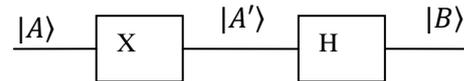
Or using matrix realization of the states

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Input A	Out1 A'	Out2 B
$ 0\rangle$	$ 1\rangle$	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$
$ 1\rangle$	$ 0\rangle$	$\frac{1}{\sqrt{2}}(0\rangle + 1\rangle)$

Truth table of the circuit



Input output relation can be written as:

$$|B\rangle = T|A\rangle$$

Where the transfer matrix T is given by:

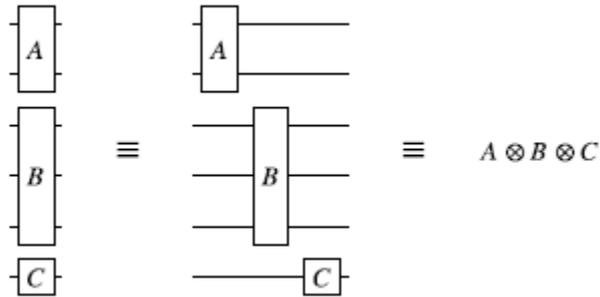
$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

The same matrix can also be obtained from the matrix product:

$$T = H \cdot X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

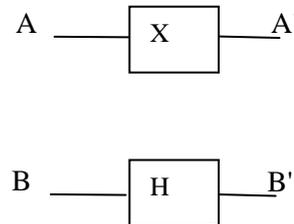
Composing quantum gates in parallel

Consider the adjacent gates in a circuit acting on a subset of a qubit as in the figure. Net effect of the gates on the qubits can be obtained by direct product of the matrix representations of the qubits. One can also obtain transfer matrix using truth table.



Direct product of the matrix was defined in the previous section. Consider the following circuit. Truth table of the circuit is also obtained using matrix representations of the gates. Transfer matrix from the truth table can be obtained as follows:

Input		Output	
$ A\rangle$	$ B\rangle$	$ A'\rangle$	$ B'\rangle$
$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$\frac{1}{\sqrt{2}}(0\rangle + 1\rangle)$
$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$
$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$\frac{1}{\sqrt{2}}(0\rangle + 1\rangle)$
$ 1\rangle$	$ 1\rangle$	$ 0\rangle$	$\frac{1}{\sqrt{2}}(0\rangle - 1\rangle)$



Truth Table. Linear combination of qubits are possible.

Part of a circuit. Two parallel quantum inverter. The inputs are qubits (0,1)

Important!!

Truth table read as follows:

The inputs A and B represent $|A\rangle|B\rangle = |AB\rangle$

The outputs A and B represent $|A'\rangle|B'\rangle = |A'B'\rangle$

Explicitly

$$\begin{aligned} |00\rangle &\rightarrow |11\rangle \\ |01\rangle &\rightarrow |10\rangle \\ |10\rangle &\rightarrow |01\rangle \\ |11\rangle &\rightarrow |00\rangle \end{aligned}$$

Important!!

Truth table read as follows:

The inputs A and B represent $|A\rangle|B\rangle = |AB\rangle$

The outputs A and B represent $|A'\rangle|B'\rangle = |A'B'\rangle$

Explicitly

$$\begin{aligned} |00\rangle &\rightarrow \frac{1}{\sqrt{2}}|1\rangle(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle) \\ |01\rangle &\rightarrow \frac{1}{\sqrt{2}}|1\rangle(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle) \end{aligned}$$

$$|10\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

$$|11\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$$

In the matrix form they can be written as:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

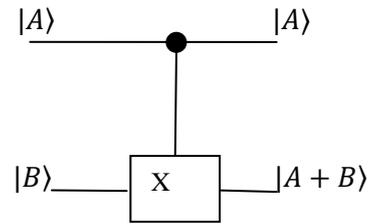
Then the transfer matrix can be written as:

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} = X \otimes H$$

Controlled quantum Gates

Sometimes we need to compute effect of a gate that is applied conditionally. The gates that implement these “if-then-else” type operations are called *controlled* gates. Note that controlled quantum Gates are more sophisticated than classical Gates. Consider the following controlled NOT (CNOT) gate.

Input		Output	
$ A\rangle$	$ B\rangle$	$ A\rangle$	$ A + B\rangle$
$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$
$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$
$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$
$ 1\rangle$	$ 1\rangle$	$ 1\rangle$	$ 0\rangle$



Truth Table. Linear combination of qbits are possible.

A controlled gate CNOT

Important!!

Truth table read as follows:

The inputs A and B represent $|A\rangle|B\rangle = |AB\rangle$

The outputs A and B represent $|A\rangle|A + B\rangle$,

A+B is binary sum of the inputs.

Explicitly

$$\begin{aligned}
 |00\rangle &\rightarrow |00\rangle \\
 |01\rangle &\rightarrow |01\rangle \\
 |10\rangle &\rightarrow |11\rangle \\
 |11\rangle &\rightarrow |10\rangle
 \end{aligned}$$

Using truth table we can write the output of the circuit:

$$\begin{aligned}
 |00\rangle &\rightarrow |00\rangle \\
 |01\rangle &\rightarrow |01\rangle \\
 |10\rangle &\rightarrow |11\rangle \\
 |11\rangle &\rightarrow |10\rangle
 \end{aligned}$$

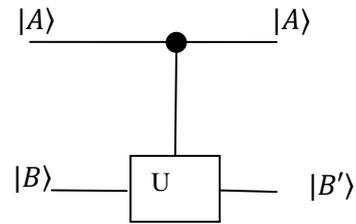
This leads to transfer matrix of the CNOT gate:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It is obvious that T is direct sum of 2x2 identity matrix and X matrix,

$$T = I_2 \oplus X$$

Input		Output	
$ A\rangle$	$ B\rangle$	$ A\rangle$	$ B'\rangle$
$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$
$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$
$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$u_{11} 0\rangle + u_{12} 1\rangle$
$ 1\rangle$	$ 1\rangle$	$ 1\rangle$	$u_{21} 0\rangle + u_{22} 1\rangle$



Truth Table. Linear combination of qbits are possible.

Controlled U-gate.
($|1\rangle$ controlled gate)

Important!!
 Truth table read as follows:
 The inputs A and B represent $|A\rangle|B\rangle = |AB\rangle$
 Explicitly

$$\begin{aligned}
 |00\rangle &\rightarrow |00\rangle \\
 |01\rangle &\rightarrow |10\rangle \\
 |10\rangle &\rightarrow u_{11}|10\rangle + u_{12}|11\rangle \\
 |11\rangle &\rightarrow u_{21}|10\rangle + u_{22}|11\rangle
 \end{aligned}$$

In general for the circuit given in the figure we can write transfer matrix

$$T = I_2 \oplus U$$

Where U is 2x2 unitary matrix.

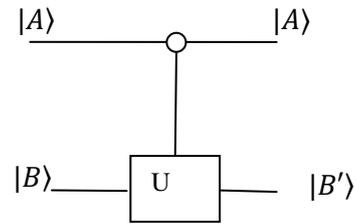
Explicitly T can be expressed as:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{pmatrix}$$

The other controlled gate perform different controlled action is given in the figure. Matrix representation of this controlled gate is

$$T = \begin{pmatrix} u_{11} & u_{12} & 0 & 0 \\ u_{21} & u_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Input		Output	
$ A\rangle$	$ B\rangle$	$ A\rangle$	$ B'\rangle$
$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$u_{11} 0\rangle + u_{12} 1\rangle$
$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$u_{21} 0\rangle + u_{22} 1\rangle$
$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$
$ 1\rangle$	$ 1\rangle$	$ 1\rangle$	$ 1\rangle$



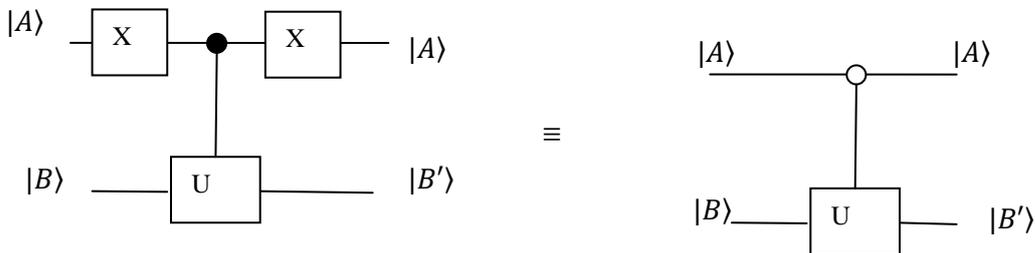
Truth Table. Linear combination of qbits are possible.

Controlled U-gate. ($|0\rangle$ controlled gate)

Important!!
 Truth table read as follows:
 The inputs A and B represent $|A\rangle|B\rangle = |AB\rangle$
 Explicitly

$$\begin{aligned}
 |00\rangle &\rightarrow u_{11}|00\rangle + u_{12}|01\rangle \\
 |01\rangle &\rightarrow u_{21}|00\rangle + u_{22}|01\rangle \\
 |10\rangle &\rightarrow |10\rangle \\
 |11\rangle &\rightarrow |11\rangle
 \end{aligned}$$

Equivalent circuit of $|0\rangle$ controlled gate can also be drawn in the figure



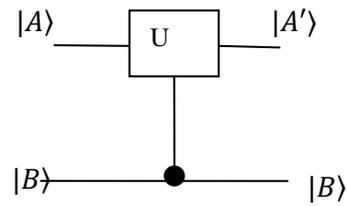
Equivalence of controlled gates Controlled U-gates.

Matrix representation of the inverted controlled gates can be obtained from the truth table. Consider the inverted controlled gate in the figure. It is $|1\rangle$ controlled gate. In this case $|B'\rangle$ is copy of $|B\rangle$. Matrix representation of the gate is given by

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{12} \\ 0 & 0 & 1 & 0 \\ 0 & u_{21} & 0 & u_{22} \end{pmatrix}$$

Exercise: Determine transfer matrix of the inverted $|0\rangle$ controlled U-gate.

Input		Output	
$ A\rangle$	$ B\rangle$	$ A\rangle$	$ B'\rangle$
$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$
$ 0\rangle$	$ 1\rangle$	$u_{11} 0\rangle + u_{12} 1\rangle$	$ 1\rangle$
$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$
$ 1\rangle$	$ 1\rangle$	$u_{21} 0\rangle + u_{22} 1\rangle$	$ 1\rangle$



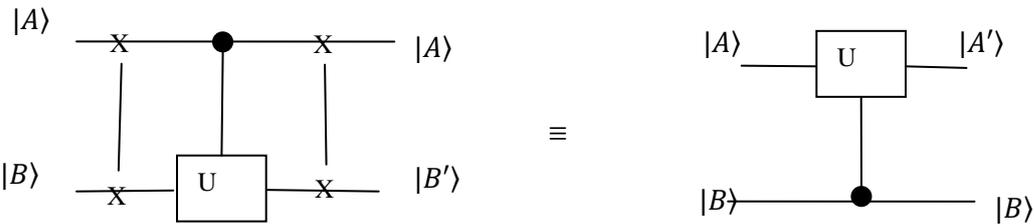
Truth Table. Linear combination of qbits are possible.

Controlled U-gate.
($|1\rangle$ controlled gate)

Important!!
 Truth table read as follows:
 The inputs A and B represent $|A\rangle|B\rangle = |AB\rangle$
 Explicitly

$$\begin{aligned}
 |00\rangle &\rightarrow |00\rangle \\
 |01\rangle &\rightarrow u_{11}|01\rangle + u_{12}|11\rangle \\
 |10\rangle &\rightarrow |10\rangle \\
 |11\rangle &\rightarrow u_{21}|01\rangle + u_{22}|11\rangle
 \end{aligned}$$

Circuit equivalence of the inverted controlled U-gate is given by:



Equivalence of controlled gates Controlled U-gates. The swap gates invert the controlled U gates.

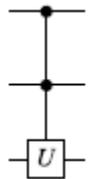
Exercise: Obtain transfer matrices of the circuits in the figure and show that the circuits are equivalent to each others.

Controlled controlled gates

Matrix representation of the controlled controlled U gate is given by

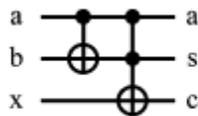
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{21} & u_{22} \end{pmatrix}$$

and its icon is



Circuit examples

1. Determine output of the half adder circuit.



Solution: The first CNOT corresponds to the matrix:

$$CN1 = CNOT \otimes I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

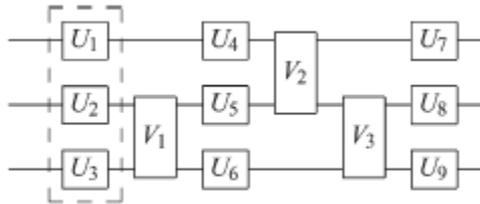
second part of the circuit is controlled controlled not gate:

$$CN2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Then transfer matrix can be obtained by using dot product,

$$CN2.CN1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

2. In the circuit U_i represent unitary matrix representations of the quantum gates. Obtain transfer matrix for the circuit.

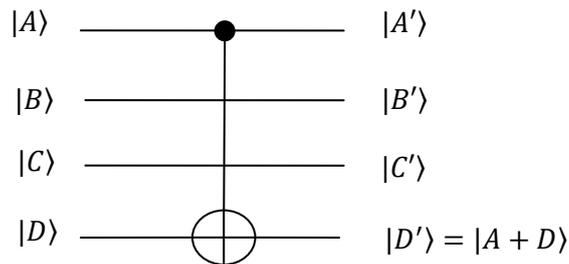


Solution:

In order to obtain output of the circuit the following operation will be presented,

$$(U_7 \otimes U_8 \otimes U_9)(I_2 \otimes V_3)(V_2 \otimes I_2)(U_4 \otimes U_5 \otimes U_6)(I_2 \otimes V_1)(U_1 \otimes U_2 \otimes U_3)$$

3. Obtain transfer matrix of the long range CNOT gate.



Solution.

Let us present truth table for the given gate

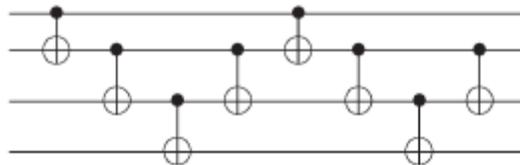
Input				Output			
$ A\rangle$	$ B\rangle$	$ C\rangle$	$ D\rangle$	$ A'\rangle$	$ B'\rangle$	$ C'\rangle$	$ D'\rangle$
$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$				
$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$
$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$
$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$
$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$
$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$
$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$ 0\rangle$

0⟩	1⟩	1⟩	1⟩	0⟩	1⟩	1⟩	1⟩
1⟩	0⟩	0⟩	0⟩	1⟩	0⟩	0⟩	1⟩
1⟩	0⟩	0⟩	1⟩	1⟩	0⟩	0⟩	0⟩
1⟩	0⟩	1⟩	0⟩	1⟩	0⟩	1⟩	1⟩
1⟩	0⟩	1⟩	1⟩	1⟩	0⟩	1⟩	0⟩
1⟩	1⟩	0⟩	0⟩	1⟩	1⟩	0⟩	1⟩
1⟩	1⟩	0⟩	1⟩	1⟩	1⟩	0⟩	0⟩
1⟩	1⟩	1⟩	0⟩	1⟩	1⟩	1⟩	1⟩
1⟩	1⟩	1⟩	1⟩	1⟩	1⟩	1⟩	0⟩

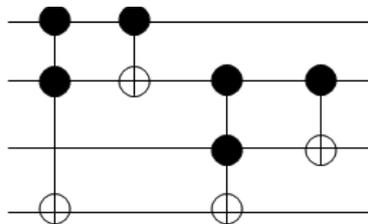
Output of the circuit is

$$|A'\rangle|B'\rangle|C'\rangle|D'\rangle = |A'B'C'D'\rangle$$

4. Obtain transfer matrix of the combination of the CNOT gates.



5. Determine transfer matrix for the circuit.



6. Determine output of the circuit.



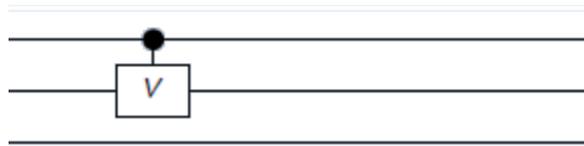
where

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Ans:

$$\begin{bmatrix} \alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta & 0 & 0 & 0 & 0 \\ \gamma & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & \beta \\ 0 & 0 & 0 & 0 & \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & 0 & \delta \end{bmatrix}$$

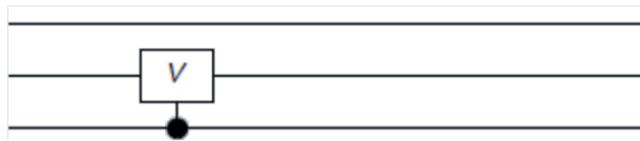
7. Determine characteristic matrix (transfer matrix) of the circuit.



Ans.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & \beta \\ 0 & 0 & 0 & 0 & \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & 0 & \delta \end{bmatrix}$$

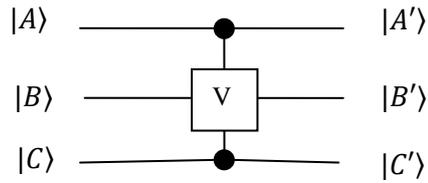
7. Determine transfer matrix of the circuit.



Ans.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & 0 & \delta \end{bmatrix}$$

8. Determine transfer matrix of the circuit.



Solution.

This controlled gate can be analysed as follows: If both $|A\rangle$ and $|C\rangle$ is $|1\rangle$ then we calculate effect of V on qubits.

Let us present truth table for the given gate

Input			Output		
$ A\rangle$	$ B\rangle$	$ C\rangle$	$ A'\rangle$	$ B'\rangle$	$ C'\rangle$
$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$	$ 0\rangle$
$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$
$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$
$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$
$ 1\rangle$	$ 0\rangle$	$ 0\rangle$	$ 1\rangle$	$ 0\rangle$	$ 0\rangle$
$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$\alpha 0\rangle + \beta 1\rangle$	$ 1\rangle$
$ 1\rangle$	$ 1\rangle$	$ 0\rangle$	$ 1\rangle$	$ 1\rangle$	$ 0\rangle$
$ 1\rangle$	$ 1\rangle$	$ 1\rangle$	$ 1\rangle$	$\gamma 0\rangle + \delta 1\rangle$	$ 1\rangle$

It can be written in the matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & \delta \end{bmatrix}$$

9. Determine transfer matrices of half and full adder circuits:

