Chapter 3. Quantum Gates

In the previous chapter we have discussed basic quantum mechanical postulates and we have shown that it is possible to construct quantum gates, such as, X-gate, H-gate etc.

Circuit model of computer is very useful for computing processes and usually used to design and construct computing hardware. In the circuit model, computer scientists use different types of Boolean logic gates acting on some binary input in order to solve various problems by composing the Gates.

In this chapter we will discuss how the notions of logic gates need to be modified in the quantum context and how they are used in the solution of the problems.

Quantum Gates

Modern computers are built using logic gates. In a similar way quantum computation also uses logic gates. The logic gates can be designed by considering unitary transformation of the qubits (states). Therefore one can construct infinitely many quantum logic gates. Quantum gates can be represented by unitary matrices. All of the quantum logic gates are reversible. No we will consider some of the gates.

In the previous chapter we have discussed the gates are known as one qubit gates:

1 qubit gates

Pauli X-Gate \( \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) or NOT gate

\[
X |0\rangle = |1\rangle, \quad X |1\rangle = |0\rangle
\]

Pauli Y-Gate \( \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \)

\[
Y |0\rangle = i |1\rangle, \quad Y |1\rangle = -i |0\rangle
\]

Pauli Z-Gate \( \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

\[
Z |0\rangle = |0\rangle, \quad Z |1\rangle = -|1\rangle
\]

Walsh Hadamard H-Gate \( H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \)
One of the simplest non-classical gate is the fractional power of NOT gate. Consider square root of Pauli-$X$ matrix:

$$\sqrt{\sigma_x} = \begin{pmatrix} 0 & 1 \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} i & \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \end{pmatrix}$$

Exercise: What is the output of the following circuits?

Rotation Gates

A single qubit pure state is represented by a point on the surface of a sphere is known as the Bloch sphere as in the figure. Bloch sphere shows computational basis states $|0\rangle$ and $|1\rangle$ and general qubit state $|\psi\rangle$. Without further discussion (in the class I will show its detail) we can introduce the following rotational matrices that represent by the rotational gates.

$$R_x(\alpha) = \exp(-i\alpha X/2) = \begin{pmatrix} \cos(\frac{\alpha}{2}) & -i\sin(\frac{\alpha}{2}) \\ -i\sin(\frac{\alpha}{2}) & \cos(\frac{\alpha}{2}) \end{pmatrix}$$

$$R_y(\alpha) = \exp(-i\alpha Y/2) = \begin{pmatrix} \cos(\frac{\alpha}{2}) & -\sin(\frac{\alpha}{2}) \\ \sin(\frac{\alpha}{2}) & \cos(\frac{\alpha}{2}) \end{pmatrix}$$

$$R_z(\alpha) = \exp(-i\alpha Z/2) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$$

$$P\theta(\delta) = e^{i\delta} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
Note that X,Y,Z and Hadmard gates can be obtained from rotation gates.

For example

\[ H = R_y \left( \frac{\pi}{2} \right) R_z (\pi) P h \left( \frac{\pi}{2} \right) \]

\[ \sqrt{\text{NOT}} = R_x \left( \frac{\pi}{2} \right) P h \left( \frac{\pi}{4} \right). \]

Controlled quantum gates

The gates implemented IF-THEN-ELSE type operations are called CONTROLLED GATEs.

We introduce some new gates. Some of these are CNOT (controlled-NOT), FREDKİN (controlled SWAP), TOFFOLI (controlled-controlled-NOT).

CNOT Gate

Icon of the gate is shown in the figure.
The truth table for the inputs as classical bits are

<table>
<thead>
<tr>
<th>input</th>
<th>output</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>(x ⊕ y)</td>
</tr>
<tr>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>0 1</td>
<td>0 1</td>
</tr>
<tr>
<td>1 0</td>
<td>1 1</td>
</tr>
<tr>
<td>1 1</td>
<td>1 0</td>
</tr>
</tbody>
</table>

The matrix of the CNOT can be obtained from the transformation

\[
\begin{align*}
|00\rangle & \xrightarrow{\text{CNOT}} |00\rangle \\
|01\rangle & \xrightarrow{\text{CNOT}} |01\rangle \\
|10\rangle & \xrightarrow{\text{CNOT}} |11\rangle \\
|11\rangle & \xrightarrow{\text{CNOT}} |10\rangle \\
\end{align*}
\]

\[
\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

Note that CNOT gate can be regarded as a 1 bit copy machine. Let us try to make a copy of the input \(|C_{in}\rangle = a|0\rangle + b|1\rangle\). The state \(|D_{in}\rangle = |0\rangle\) and joint state \(|C_{in}\rangle \otimes |D_{in}\rangle = a|00\rangle + b|10\rangle\). The output state \(|D_{out}\rangle = |C_{in}\rangle \otimes |D_{in}\rangle = a|00\rangle + b|11\rangle\). This is an entangled pair. May be, our mood is a bit ambivalent but later (next chapters) we will discuss the entanglement.

SWAP Gate

The icon for the SWAP gate is
Troth table of the SWAP gate is given by

<table>
<thead>
<tr>
<th>input</th>
<th>output</th>
</tr>
</thead>
<tbody>
<tr>
<td>x y</td>
<td>x' y'</td>
</tr>
<tr>
<td>0 0</td>
<td>0 0</td>
</tr>
<tr>
<td>0 1</td>
<td>1 0</td>
</tr>
<tr>
<td>1 0</td>
<td>0 1</td>
</tr>
<tr>
<td>1 1</td>
<td>1 1</td>
</tr>
</tbody>
</table>

FREDKIN gate

The icon for the FREDKIN gate as shown in the figure:

Truth table of the fredkin gate are given by

<table>
<thead>
<tr>
<th>input</th>
<th>output</th>
</tr>
</thead>
<tbody>
<tr>
<td>x y z</td>
<td>x' y' z'</td>
</tr>
<tr>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
<tr>
<td>0 0 1</td>
<td>0 0 1</td>
</tr>
<tr>
<td>0 1 0</td>
<td>0 1 0</td>
</tr>
</tbody>
</table>
Transformation of the state by using FREDKIN gate is given by:

\[
\begin{array}{c|c}
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\]

Transformation of the state by using FREDKIN gate is given by:

\[
\begin{align*}
|011\rangle & \xrightarrow{\text{FREDKIN}} |011\rangle \\
|100\rangle & \xrightarrow{\text{FREDKIN}} |100\rangle \\
|101\rangle & \xrightarrow{\text{FREDKIN}} |110\rangle \\
|110\rangle & \xrightarrow{\text{FREDKIN}} |101\rangle \\
|111\rangle & \xrightarrow{\text{FREDKIN}} |111\rangle \\
\end{align*}
\]

Matrix representation of the FREDKIN gate can be easily obtained from the transformation.

TOFFOLI Gate

Icon of the TOFFOLI gate is given by

\[
\begin{align*}
|A_{in}\rangle & \quad x \\
|B_{in}\rangle & \quad y \\
|C_{in}\rangle & \quad z \\
\end{align*}
\]

\[
\begin{align*}
|A_{out}\rangle & \\
|B_{out}\rangle & \\
|C_{out}\rangle & \\
\end{align*}
\]

Transformation of the states using TOFFOLI gate are given by:
We can set up a many gates by using quantum mechanical unitary transformation. Some of the well known gates are NOR, NAND, NMAJORITY, DEUTSCH, BARENCO, CSIGN, B Berkeley etc. One can obtain an interrelation between the various gates.